PT-symmetric quantum field theory

For Christmas I want a DRAGON!

Be realistic.

A unified quantum mechanical theory of gravitation.

What color do you want your DRAGON.

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What is the positive root of $x^5 + x = 1$?

(It happens to be $x = 0.75488...$)

Insert $\varepsilon$: $x^5 + \varepsilon x = 1$ (strong coupling)

$$x(\varepsilon) = \sum_{n=0}^{\infty} a_n \varepsilon^n \quad a_0 = 1$$
Match powers of $\varepsilon$

\[ 5a_1 + 1 = 0, \]
\[ 5a_2 + 10a_1^2 + a_1 = 0, \]
\[ 5a_3 + 20a_1a_2 + a_2 + 10a_1^3 = 0, \]
\[ 5a_4 + 20a_1a_3 + a_3 + 10a_2^2 + 30a_1^2a_2 + 5a_1^4 = 0. \]

\[ a_1 = -\frac{1}{5}, \quad a_2 = -\frac{1}{25}, \quad a_3 = -\frac{1}{125}, \]
\[ a_4 = 0, \quad a_5 = \frac{21}{15625}, \quad a_6 = \frac{78}{78125}. \]
The perturbation series...

\[ x(\epsilon) = 1 - \frac{1}{5}\epsilon - \frac{1}{25}\epsilon^2 - \frac{1}{125}\epsilon^3 + \frac{21}{15625}\epsilon^5 + \frac{78}{78125}\epsilon^6 + \cdots \]

Sum the series at \( \epsilon = 1 \)

(Luckily, radius of convergence of series = \( \frac{5}{4^{4/5}} = 1.64938\ldots \))

Sixth-order result: \( x(1) = 0.75434 \)

Exact answer: \( x = 0.75488 \)

(0.07\% error)
The physicist’s way to insert $\varepsilon$

Insert $\varepsilon$:
\[ \varepsilon x^5 + x = 1 \] (weak coupling)

Perturbation series:
\[ x(\varepsilon) = 1 - \varepsilon + 5\varepsilon^2 - 35\varepsilon^3 + 285\varepsilon^4 - 2530\varepsilon^5 + 23751\varepsilon^6 - \ldots \]

Sum the series at $\varepsilon = 1$
(Unfortunately, radius of convergence of series $= 4^4/5^5 = 0.08192$)

Result: $x(1) = 21476$

Fortunately, (3,3)-Padé gives 0.7% accuracy (not bad!)
Yet another way to insert $\varepsilon \to \delta$

$$x^{1+\delta} + x = 1$$

Perturbation series (called the \textit{delta expansion}): \quad x(\delta) = c_0 + c_1 \delta + c_2 \delta^2 + c_3 \delta^3 \ldots$$

$$c_0 = \frac{1}{2}, \quad c_1 = \frac{1}{4} \ln 2, \quad c_2 = -\frac{1}{8} \ln 2,$$
$$c_3 = -\frac{1}{48} \ln^3 2 + \frac{1}{8} \ln^2 2 + \frac{1}{16} \ln 2,$$
$$c_4 = \frac{1}{8} \ln^3 2 - \frac{5}{16} \ln^2 2 - \frac{1}{32} \ln 2,$$
$$c_5 = \frac{1}{480} \ln^5 2 - \frac{7}{768} \ln^4 2 - \frac{3}{128} \ln^3 2 + \frac{3}{64} \ln^2 2$$
$$+ \frac{1}{64} \ln 2,$$
$$c_6 = -\frac{1}{192} \ln^5 2 + \frac{35}{1536} \ln^4 2 + \frac{5}{768} \ln^3 2 - \frac{5}{128} \ln^2 2$$
$$- \frac{1}{128} \ln 2$$

Radius of convergence of this series = 1

We must sum the series at $\delta = 4$

(3,3)-Padé gives 0.5% accuracy – \textit{better than weak coupling}!
Thomas-Fermi equation (nuclear structure)

\[ y''(x) = \frac{[y(x)]^{3/2}}{\sqrt{x}} \]
\[ y(0) = 1, y(\infty) = 0 \]

Delta expansion:
\[ y''(x) = y(x)\left[\frac{y(x)}{x}\right]^\delta \quad y(0) = 1, y(\infty) = 0 \]
\[ y(x) = y_0(x) + \delta y_1(x) + \delta^2 y_2(x) + \cdots \]

Third-order calculation with (2,1)-Padé approximant gives 1.1% accuracy!
Blasius Equation  (fluid mechanics)

\[ y'''(x) + y''(x)y(x) = 0 \]
\[ y(0) = y'(0) = 0, \quad y'(\infty) = 1 \]

\[ y'''(x) + y''(x)[y(x)]^\delta = 0 \]
\[ y(0) = y'(0) = 0, \quad y'(\infty) = 1 \]

\[ y(x) = y_0(x) + \delta y_1(x) + \delta^2 y_2(x) + \cdots \]

Second-order calculation with 
(1,1)-Padé approximant gives 8.7% accuracy!
Lane-Emden equation (stellar structure)

\[ y''(x) + 2y'(x)/x + [y(x)]^n = 0 \]

\[ y''(x) + 2y'(x) + [y(x)]^{1+\delta} = 0 \]

Korteweg-de Vries equation (nonlinear waves)

\[ u_t + uu_x + u_{xxx} = 0 \]

\[ u_t + u^\delta u_x + u_{xxx} = 0 \]
THE MESSAGE OF THIS TALK:

The $\delta$ expansion can be applied in quantum mechanics --- and it can be extended to quantum field theory!!

You’ll like this talk; please turn off your computer and listen!
What about *quantum mechanics*?

Replace

\[ H = p^2 + x^4 \]

with

\[ H = p^2 + x^2 \left( x^2 \right)^\delta \]

and expand in powers of \( \delta \), but note that we want to raise a *positive* quantity to the power \( \delta \) to avoid complex numbers appearing as an artefact of the procedure.

This suggests an interesting idea:

\[ H = p^2 + x^2 \left( ix \right)^\varepsilon \]

This Hamiltonian is not Hermitian but is *PT* symmetric.
**PT**-symmetric quantum mechanics: Extending quantum mechanics into the **complex** domain…

If you respect **PT** symmetry, the eigenvalues can remain real and unitarity can be preserved even though the Hamiltonian is not Hermitian.

**PT** reflection – a *simultaneous reflection of space and time* (related to the **CPT** theorem)
Space-time reflection
$PT$-symmetric quantum theory is a highly active area of research!

- Nearly 4,000 published papers and many books
- Scores of theses written
- Scores of international conferences and symposia entirely devoted to $PT$ symmetry
EXAMPLE: 1-parameter family of $PT$-symmetric Hamiltonians obtained by complex deformation of the harmonic oscillator

$$H = p^2 + x^2 (i x)^\varepsilon \quad (\varepsilon \text{ real})$$

Look! $H$ is not Hermitian but its eigenvalues are all real!

Special cases:  

Cubic: $H = p^2 + i x^3 \ (\varepsilon = 1)$  

Quartic: $H = p^2 - x^4 \ (\varepsilon = 2)$

P. Dorey, C. Dunning, and R. Tateo  

P. Dorey, C. Dunning, and R. Tateo  
PT-symmetric Hamiltonians obey the axioms of QM, but are *complex generalizations (deformations)* of Hermitian Hamiltonians.

You begin with a Hermitian Hamiltonian and introduce a *deformation parameter* $\varepsilon$ …

Complex deformed squirrel  
Complex deformed frog  
Complex deformed parrot
Simple example: Complex deformed harmonic oscillator

\[ H = p^2 + x^2 + i\varepsilon x \]

\[-\psi''(x) + x^2 \psi(x) + i\varepsilon x \psi(x) = E \psi(x) \quad [\psi(\pm\infty) = 0]\]

\[ E_n = 2n + 1 + \varepsilon^2/4 \quad (n = 0, 1, 2, 3, \ldots)\]
PT SYMMETRY IN QUANTUM PHYSICS:
FROM A MATHEMATICAL CURIOSITY
TO OPTICAL EXPERIMENTS

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Space-time reflection symmetry, or PT symmetry, first proposed in quantum mechanics by Bender and Boettcher in 1998 [1], has become an active research area in fundamental physics. More than two thousand papers have been published on the subject and papers have appeared in two dozen categories of the arXiv. Over two dozen international conferences and symposia specifically devoted to PT symmetry have been held and many PhD theses have been written.
Classical PT symmetry:

\[ H = p^2 + i x^3 \quad (\varepsilon = 1) \]

Classical trajectories in the complex-\(x\) plane

Average location of a classical particle is a pure negative imaginary number. In quantum mechanics \(\langle x \rangle\) is also negative imaginary. In quantum field theory the 1-point Green’s function, say for a \(-\phi^4\) field theory (\(\varepsilon = 2\)), is also negative imaginary.
Theoretical applications: renormalizing makes a Hamiltonian non-Hermitian, but it becomes $PT$ symmetric

- Lee model is unitary (there are no ghosts!)
- Pais-Uhlenbeck model (no ghosts!)
- Self-force on the electron (runaway modes)
- Double-scaling limit in QFT
- Stability of the Higgs vacuum
- Liouville quantum field theory (Curtright!)
- Conformal gravity (Mannheim!)
- Asymptotic behavior of the Painlevé transcendent
- Application to the Riemann hypothesis
  …and many many many many more!
Experimental Studies of $PT$ symmetry:

- $PT$-symmetric wave guides
- $PT$-symmetric lasers
- $PT$-symmetric electronic and mechanical systems
- Unidirectional transmission of light
- $PT$-symmetric atomic diffusion
- $PT$-symmetric superconducting wires
- $PT$-symmetric optical graphene
- $PT$-symmetric topological insulators

…and many many many many more!
From the article:

Top 10 physics discoveries of the last 10 years

Jorge Cham

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First observation of $PT$ transition using optical wave guides

Observation of parity-time symmetry in optics

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One of the fundamental axioms of quantum mechanics is associated with the Hermiticity of physical observables. In the case of the Hamiltonian operator, this requirement not only implies real eigenenergies but also guarantees probability conservation. Interestingly, a wide class of non-Hermitian Hamiltonians can still show entirely real spectra. Among these are Hamiltonians respecting parity-time (PT) symmetry. Even though the Hermiticity of quantum observables was never in doubt, such concepts have motivated discussions on several fronts in physics, including quantum field theories, non-Hermitian Anderson models and open quantum systems, to mention a few. Although the impact of PT symmetry in these fields is still debated, it has been recently realized that optics can provide a fertile ground where PT-related notions can be implemented and experimentally investigated. In this letter we report the first observation of the behaviour of a PT optical coupled system that judiciously involves a complex index potential. We observe both spontaneous PT symmetry breaking and power oscillations violating left-right symmetry. Our results may pave the way towards a new class of PT-synthetic materials with intriguing and unexpected properties that rely on non-reciprocal light propagation and tailored transverse energy flow.

(ε > εth), the spectrum ceases to be real and starts to involve imaginary eigenvalues. This signifies the onset of a spontaneous PT symmetry-breaking, that is, a 'phase transition' from the exact to broken-PT phase. In optics, several physical processes are known to obey equations that are formally equivalent to that of Schrödinger in quantum mechanics. Spatial diffraction and temporal dispersion are perhaps the most prominent examples. In this work we focus our attention on the spatial domain, for example optical beam propagation in PT-symmetric complex potentials. In fact, such PT 'optical potentials' can be realized through a judicious inclusion of index guiding and gain/loss regions. Given that the complex refractive-index distribution \( n(x) = n_r(x) + in_i(x) \) plays the role of an optical potential, we can then design a PT-symmetric system by satisfying the conditions \( n_r(x) = n_r(-x) \) and \( n_i(x) = -n_i(-x) \).

In other words, the refractive-index profile must be an even function of position \( x \) whereas the gain/loss distribution should be odd. Under these conditions, the electric-field envelope \( E \) of the optical beam is governed by the paraxial equation of diffraction:

\[
\frac{\partial E}{\partial z} + \frac{1}{2k} \frac{\partial^2 E}{\partial x^2} + k_0 [n_r(x) + in_i(x)] E = 0
\]
Exceptional Contours and Band Structure Design in Parity-Time
Symmetric Photonic Crystals

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We investigate the properties of two-dimensional parity-time symmetric periodic systems whose non-Hermitian periodicity is an integer multiple of the underlying Hermitian system’s periodicity. This creates a natural set of degeneracies that can undergo thresholdless $\mathcal{PT}$ transitions. We derive a $\mathbf{k} \cdot \mathbf{p}$ perturbation theory suited to the continuous eigenvalues of such systems in terms of the modes of the underlying Hermitian system. In photonic crystals, such thresholdless $\mathcal{PT}$ transitions are shown to yield significant control over the band structure of the system, and can result in all-angle supercollimation, a $\mathcal{PT}$-superprism effect, and unidirectional behavior.

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Robust wireless power transfer using a nonlinear parity–time–symmetric circuit

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Considerable progress in wireless power transfer has been made in the realm of non-radiative transfer, which employs magnetic-field coupling in the near field¹–⁴. A combination of circuit resonance and impedance transformation is often used to help to achieve efficient transfer of power over a predetermined distance of about the size of the resonators³,⁴. The development of non-radiative wireless power transfer has paved the way towards real-world applications such as wireless powering of implantable medical devices and wireless charging of stationary electric vehicles¹,²,⁵–⁸. However, it remains a fundamental challenge to create a wireless power transfer system in which the transfer efficiency is robust against the variation of operating conditions. Here we propose theoretically and demonstrate experimentally that a parity–time–symmetric circuit incorporating a nonlinear gain saturation element provides robust wireless power transfer. Our results show that the transfer efficiency remains near unity over a distance variation of approximately one metre, without the need for any tuning. This is in contrast with conventional methods where high transfer efficiency can only be maintained by constantly tuning the frequency or the internal coupling parameters as the transfer distance or the relative orientation of the source and receiver units is varied. The use of a nonlinear parity–time-symmetric circuit should enable robust wireless power transfer to moving devices or vehicles⁹,¹⁰.
$PT$-symmetric quantum field theory
(with S. P. Klevansky, N. Hassanpour, and S. Sarkar)

$D$-dimensional Euclidean-space quantum field theory
with a pseudoscalar field

\[ \mathcal{L} = \frac{1}{2}(\partial \phi)^2 + \frac{1}{2} \phi^2 (i \phi)^\varepsilon \quad (\varepsilon \geq 0) \]

To avoid renormalization infinities assume $D < 2$.

**Objective:** Calculate the vacuum energy density, the renormalized mass, and the Green’s functions such as

\[ G_1 \]
\[ G_2(x - y) \]

as series in powers of $\varepsilon$
If we expand in $\varepsilon$ we get logarithmic terms in the Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}\phi^2 + \frac{1}{2}\varepsilon\phi^2 \log(i\phi) + O(\varepsilon^2)$$

The unperturbed Lagrangian is the usual free theory:

$$\mathcal{L}_0 = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}\phi^2$$

How do we interpret the logarithm term $\log(i\phi)$ ??
\[ \log(i\phi) = \frac{1}{2}i\pi + \log(\phi) \quad (\phi > 0) \]

\[ \log(i\phi) = -\frac{1}{2}i\pi + \log(-\phi) \quad (\phi < 0) \]

The imaginary term is odd in \( \phi \) and the real term is even in \( \phi \) --- this is how to ensure \textit{PT} symmetry!
Free propagator in momentum space

$$\tilde{\Delta}(p) = \frac{1}{p^2 + 1}$$

and in coordinate space

$$\Delta(x_1 - x_2) = (2\pi)^{-\frac{D}{2}}|x_1 - x_2|^{1-\frac{D}{2}}K_{1-\frac{D}{2}}(|x_1 - x_2|)$$

$$\Delta(0) = (4\pi)^{-D/2}\Gamma(1 - D/2)$$
The partition function:

\[ Z(\varepsilon) = \int \mathcal{D}\phi \exp \left( - \int d^D x \mathcal{L} \right) \]

\[ Z(0) = \int \mathcal{D}\phi \exp \left( - \int d^D x \mathcal{L}_0 \right) \]

Expand \( Z(\varepsilon) \) to first order in \( \varepsilon \)

\[ Z(\varepsilon) = \int \mathcal{D}\phi e^{-\int d^D x \mathcal{L}_0} \left[ 1 - \frac{\varepsilon}{4} \int d^D y \left\{ i\pi \phi(y) |\phi(y)| + \phi^2(y) \log \left[ \phi^2(y) \right] \right\} \right] \]

Note: Imaginary part is **odd** and real part is **even**
Ground-state energy density $E_0$

$$\Delta E = \frac{\varepsilon}{4Z(0)V} \int D\phi e^{-\int d^DxL_0} \int d^Dy\phi^2(y) \log [\phi^2(y)]$$

Replica trick of Parisi: $\log A = \lim_{N \to 0} \frac{1}{N} (A^N - 1)$

We do something simpler: $A^2 \log(A^2) = \lim_{N \to 1} \frac{d}{dN} A^{2N}$

Note: Replica trick is not rigorous as it involves a continuous limit. Validity has not been proved but when one can compare with known results (in low dimension) it has always worked.
\[ \Delta E = \lim_{N \to 1} \frac{\varepsilon}{4Z(0)V} \frac{d}{dN} \int D\phi e^{-\int d^D x \mathcal{L}_0} \int d^D y \phi^{2N}(y) \]

This is just \( N \) self loops from \( y \) to \( y \)

\[ \Delta E = \lim_{N \to 1} \frac{\varepsilon}{4V} \frac{d}{dN} \int d^D y [\Delta(0)]^N (2N - 1)!! \]

Cancel \( V \):

\[ \Delta E = \lim_{N \to 1} \frac{\varepsilon}{4} \frac{d}{dN} [\Delta(0)]^N (2N - 1)!! \]

But

\[ (2N - 1)!! = 2^N \Gamma(N + \frac{1}{2}) / \sqrt{\pi} \]
\[ \Delta E = \frac{1}{4} \varepsilon \Delta(0) \left\{ \log[2\Delta(0)] + \psi\left(\frac{3}{2}\right) \right\} \]
\[ \psi\left(\frac{3}{2}\right) = 2 - \gamma - 2 \log 2 \]

*Special case* \( D = 0 \) for which \( \Delta(0) = 1 \): \[ \Delta E = \frac{\varepsilon}{4} \left[ \psi\left(\frac{3}{2}\right) + \log 2 \right] \]

*Special case* \( D = 1 \): In quantum mechanics \( \Delta E \) to leading order in \( \varepsilon \) is the expectation value of the interaction Hamiltonian

\[ H_I = \frac{1}{2} \varepsilon x^2 \log(ix) \]

in the unperturbed ground-state wave function \( \psi_0(x) = \exp\left(-\frac{1}{2}x^2\right) \)

\[ \Delta E = \frac{\varepsilon}{2} \left[ \int_{-\infty}^{\infty} dx \, e^{-x^2} x^2 \log(ix) / \int_{-\infty}^{\infty} dx \, e^{-x^2} = \frac{1}{8} \varepsilon \psi\left(\frac{3}{2}\right) \right] \]

At \( D = 1 \) \( \Delta(0) = \frac{1}{2} \)
One-point Green’s function:

\[ G_1(a) = \frac{1}{Z(0)} \int \mathcal{D}\phi \phi(a) e^{-\int d^Dx \mathcal{L}_0} \left[ -\frac{\varepsilon}{4} \int d^Dy i\pi \phi(y) |\phi(y)| \right] \]

(keeping terms that are odd in the field)

\[ \phi|\phi| = \frac{2}{\pi} \phi^2 \int_0^\infty \frac{dt}{t} \sin(t\phi) \]

\[ G_1(a) = -\frac{i\varepsilon}{2} \int_{t=0}^\infty dt \sum_{n=0}^\infty \frac{(-1)^n t^{2n}}{(2n+1)!} \int d^Dy \]

\[ \times \frac{1}{Z(0)} \int \mathcal{D}\phi e^{-\int d^Dx \mathcal{L}_0} \phi(a) [\phi(y)]^{2n+3} \]
The second line is $2n+3$ points at $y$ and one point at $a$:

$$\frac{1}{Z(0)} \int D\phi e^{-L_0 \phi(0)} [\phi(y)]^{2n+3} = (2n + 3)!! \Delta(y - a) \Delta^{n+1}(0)$$

$$\int d^D y \Delta(y - a) = 1$$

$$G_1 = -\frac{i\varepsilon}{2} \int_{t=0}^{\infty} dt \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n} \Delta^{n+1}(0)(2n + 3)!!}{(2n + 1)!}$$

$$= -\frac{i\varepsilon}{2} \int_{t=0}^{\infty} dt \Delta(0) \left[3 - \Delta(0)t^2\right] e^{-\frac{1}{2} \Delta(0)t^2}$$

$$G_1 = -i\varepsilon \sqrt{\Delta(0)\pi/2}$$

This is negative and imaginary!

(This formula is exact to order $\varepsilon$)
Check: When $D = 0$ we get
$$G_1 = -i \varepsilon \sqrt{\pi/2}.$$ When $D = 1$ we get
$$G_1 = -i \varepsilon \sqrt{\pi/4}.$$

This generalizes to higher order in $\varepsilon$ and to all Green’s functions...
\[ G_n(x_1, x_2, \ldots x_n) = -\frac{1}{2} \varepsilon (-i)^n \Gamma \left( \frac{1}{2} n - 1 \right) \times \left[ \frac{1}{2} \Delta(0) \right]^{1-n/2} \int d^D u \prod_{k=1}^{n} \Delta(x_k - u) \]
2-point Green’s function

\[ G_2(a - b) = \Delta(a - b) - \varepsilon K \int d^Dy \Delta(a - y) \Delta(y - b) \]

\[ \tilde{G}_2(p) = \frac{1}{p^2 + 1} - \varepsilon \frac{K}{(p^2 + 1)^2} + O(\varepsilon^2) \]

\[ K = \frac{1}{2} + \frac{1}{2} \Gamma'(\frac{3}{2})/\Gamma(\frac{3}{2}) + \frac{1}{2} \log[2\Delta(0)] \]
\[ = \frac{3}{2} - \frac{1}{2} \gamma + \frac{1}{2} \log \left[\frac{1}{2} \Delta(0)\right] \]
Renormalized mass

\[ \tilde{G}_2(p) = \frac{1}{p^2 + 1 + \varepsilon K + O(\varepsilon^2)} \]

\[ M_R^2 = 1 + K\varepsilon + O(\varepsilon^2) \]

\[ K = \frac{1}{2} + \frac{1}{2} \Gamma'(\frac{3}{2})/\Gamma(\frac{3}{2}) + \frac{1}{2} \log[2\Delta(0)] \]
\[ = \frac{3}{2} - \frac{1}{2} \gamma + \frac{1}{2} \log \left[\frac{1}{2} \Delta(0)\right] \]
Comments on renormalization…
for listening to my talk!