Protostring Scattering

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Introduction

20 years ago Oren Bergman and I developed a string bit model for the free superstring. But we couldn't find a natural way to produce operator insertions required of the three string vertex.

We knew our model wasn't the superstring. But only in the last few years have I tried to understand what physics our model did describe.

The answer, reported here two years ago is what I call the generalized protostring: take the bosonic string in 24 transverse dimensions, and replace s of the dimensions with Grassmann dimensions: $24 = d + s$. The model is critical but not Lorentz invariant. The limiting case $d = 0$ is my original protostring.

In this talk I would like to describe the scattering amplitudes for the protostring: arXiv:1607.04237,

Outline

1. Review String bit models
2. Bosonization
3. Amplitudes from Mandelstam's Lightcone formalism
4. Simple example and discussion
5. Conclusion

Some Simple Superstring Bit Models

- Superstring bit annihilation operators

$$(\phi_{[a_1\cdots a_n]})_\alpha^{\ \beta}; \qquad n=0,\ldots,s; \qquad a_i=1,\ldots,s; \qquad \alpha,\beta=1,\ldots,N$$
$$\bar\phi_\alpha^{\ \beta} \equiv (\phi_\beta^{\ \alpha})^\dagger, \qquad [\phi,\bar\phi]_\pm = \text{Obvious Kronecker Delta's}$$

- n even : Boson, $\qquad n$ odd : Fermion

- No transverse space; No longitudinal space; Only time

- Finite number $= 2^s N^2$ degrees of freedom.

- Supersymmetry: $\{Q_a, Q_b\} = 2M\delta_{ab}$:

$$Q_a = \sum_{n=0}^{s-1} \frac{(-)^n}{n!} \mathrm{Tr}\left[e^{i\pi/4}\bar\phi_{a_1\cdots a_n}\phi_{aa_1\cdots a_n} + e^{-i\pi/4}\bar\phi_{aa_1\cdots a_n}\phi_{a_1\cdots a_n}\right]$$

$H = \sum_{j=1}^{5} H_j$ satisfies $[Q_a, H] = 0$ with:

$$
\begin{aligned}
H_1 &= \frac{2}{N} \sum_{n=0}^{s} \sum_{k=0}^{s} \frac{s-2n}{n!k!} \mathrm{Tr} \bar{\phi}_{a_1 \cdots a_n} \bar{\phi}_{b_1 \cdots b_k} \phi_{b_1 \cdots b_k} \phi_{a_1 \cdots a_n} \\
H_2 &= \frac{2}{N} \sum_{n=0}^{s-1} \sum_{k=0}^{s-1} \frac{(-)^k}{n!k!} \mathrm{Tr} \bar{\phi}_{a_1 \cdots a_n} \bar{\phi}_{b b_1 \cdots b_k} \phi_{b_1 \cdots b_k} \phi_{b a_1 \cdots a_n} \\
H_3 &= \frac{2}{N} \sum_{n=0}^{s-1} \sum_{k=0}^{s-1} \frac{(-)^k}{n!k!} \mathrm{Tr} \bar{\phi}_{b a_1 \cdots a_n} \bar{\phi}_{b_1 \cdots b_k} \phi_{b b_1 \cdots b_k} \phi_{a_1 \cdots a_n} \\
H_4 &= \frac{2i}{N} \sum_{n=0}^{s-1} \sum_{k=0}^{s-1} \frac{(-)^k}{n!k!} \mathrm{Tr} \bar{\phi}_{a_1 \cdots a_n} \bar{\phi}_{b_1 \cdots b_k} \phi_{b b_1 \cdots b_k} \phi_{b a_1 \cdots a_n} \\
H_5 &= -\frac{2i}{N} \sum_{n=0}^{s-1} \sum_{k=0}^{s-1} \frac{(-)^k}{n!k!} \mathrm{Tr} \bar{\phi}_{b a_1 \cdots a_n} \bar{\phi}_{b b_1 \cdots b_k} \phi_{b_1 \cdots b_k} \phi_{a_1 \cdots a_n} .
\end{aligned}
$$

Hamiltonian for $s = 1$

$$\phi, \phi_1 \to a, b$$

Superstring bit Hamiltonian::

$$H = \frac{2}{N} \mathrm{Tr} \left[(\bar{a}^2 - i\bar{b}^2)a^2 - (\bar{b}^2 - i\bar{a}^2)b^2 + (\bar{a}\bar{b} + \bar{b}\bar{a})ba + (\bar{a}\bar{b} - \bar{b}\bar{a})ab \right]$$

This is ordinary QM with $2N^2$ d.o.f.

Supersymmetry:

$$Q = \mathrm{Tr}(\bar{a}be^{i\pi/4} + \bar{b}ae^{-i\pi/4}), \qquad [Q, H] = 0, \qquad Q^2 = \mathrm{Tr}\left[\bar{a}a + \bar{b}b\right] \equiv M$$

Relation to String

$$P^+ \quad = \quad mM, \qquad P^- = \frac{T_0}{4m} H$$

States with M very large are stringy if excitations of P^- have large M behavior $\sim M^{-1}$

These excitations are statistics waves which have low excited P^- because of the SUSY generated by Q.

Introduce superfield: $\Phi(\theta) = \sum_{n,a_k} \bar{\phi}_{[a_1 \cdots a_n]} \theta^{a_1} \cdots \theta^{a_n}$

$$\begin{aligned} H \int d^{sM}\theta \mathrm{Tr}\Phi(\theta_1) \cdots \Phi(\theta_M)|0\rangle \Psi(\theta_1, \ldots, \theta_M) \\ = \int d^M\theta \mathrm{Tr}\Phi(\theta_1) \cdots \Phi(\theta_M)|0\rangle h\Psi(\theta_1, \ldots, \theta_M) + \mathcal{O}(1/N) \end{aligned}$$

$N \to \infty$: First Quantized Hamiltonian

$$h_{kl} = \sum_{a=1}^{s} \left[-2i\theta_k^a \theta_l^a - 2i \frac{d}{d\theta_k^a} \frac{d}{d\theta_l^a} - 2\theta_k^a \frac{d}{d\theta_l^a} - 2\theta_l^a \frac{d}{d\theta_k^a} - 2 + 4\theta_k^a \frac{d}{d\theta_k^a} \right]$$

Then

$$h = \sum_k h_{k,k+1}$$

Clifford algebra:

$$S_k = \theta_k + \frac{d}{d\theta_k}, \qquad \tilde{S}_k = i\left(\theta_k - \frac{d}{d\theta_k}\right)$$

$$h = \sum_k \left[-iS_k S_{k+1} + i\tilde{S}_k \tilde{S}_{k+1} - iS_k \tilde{S}_{k+1} + i\tilde{S}_k S_{k+1} + 2iS_k \tilde{S}_k \right]$$

$$\{S_k, S_l\} = 2\delta_{kl}, \qquad \{\tilde{S}_k, \tilde{S}_l\} = 2\delta_{kl}, \qquad \{S_k, \tilde{S}_l\} = 0$$

Last term handles lattice fermion doubling problem.

Eigenstates of h

To find the eigenvalues of h, Fourier transform

$$B_n = \frac{1}{\sqrt{M}} \sum_{k=1}^{M} S_k e^{-2\pi i k n/M}, \qquad \tilde{B}_n = \frac{1}{\sqrt{M}} \sum_{k=1}^{M} \tilde{S}_k e^{-2\pi i k n/M}$$

$$\{B_m, B_n\} = 2\delta_{m+n,M}, \qquad \{\tilde{B}_m, \tilde{B}_n\} = 2\delta_{m+n,M}, \qquad \{\tilde{B}_m, B_n\} = 0$$

Ground Energy: $F_m|G\rangle = 0$, $m > 0$

$$P_G^- = -\frac{T_0}{m} \sum_{n=1}^{M-1} \sin\frac{n\pi}{M} \sim -\frac{2T_0 M}{m\pi} + \frac{\pi T_0}{6Mm}$$

H splits or fuses traces at order $1/N$

Critical dimension determined by demand that the three closed string vertex scale as $\sim M^{-3}$:

- Finite continuum limit

- Ensures $O(1,1)$ subgroup of the Lorentz group.

Critical dimension

Everything done for $s = 1$ extends to general s w/o difficulty!

Overlap	Scaling	Insertion	Scaling
V_X	$M^{-d/8}$	ΔX	$M^{-1/2}$
V_S	$M^{-s/8}$	S	M^0
V_Γ	$M^{-d/16}$	Γ	$M^{-1/4}$

Smooth continuum limit: vertex $\sim M^{-3}$ for $K/M, L/M$ fixed.

Type	Total Vertex	Critical Dimension
Bosonic String	V_X	$d = 24$
IIB Superstring	$\Delta X^i \Delta X^j \mathcal{P}_{ij}(S) V_X V_S$	$d = s = 8$
RNS String	$(\Gamma \cdot \Delta X)^2 V_\Gamma V_X$	$d = 8$
Protostring	V_S	$s = 24$

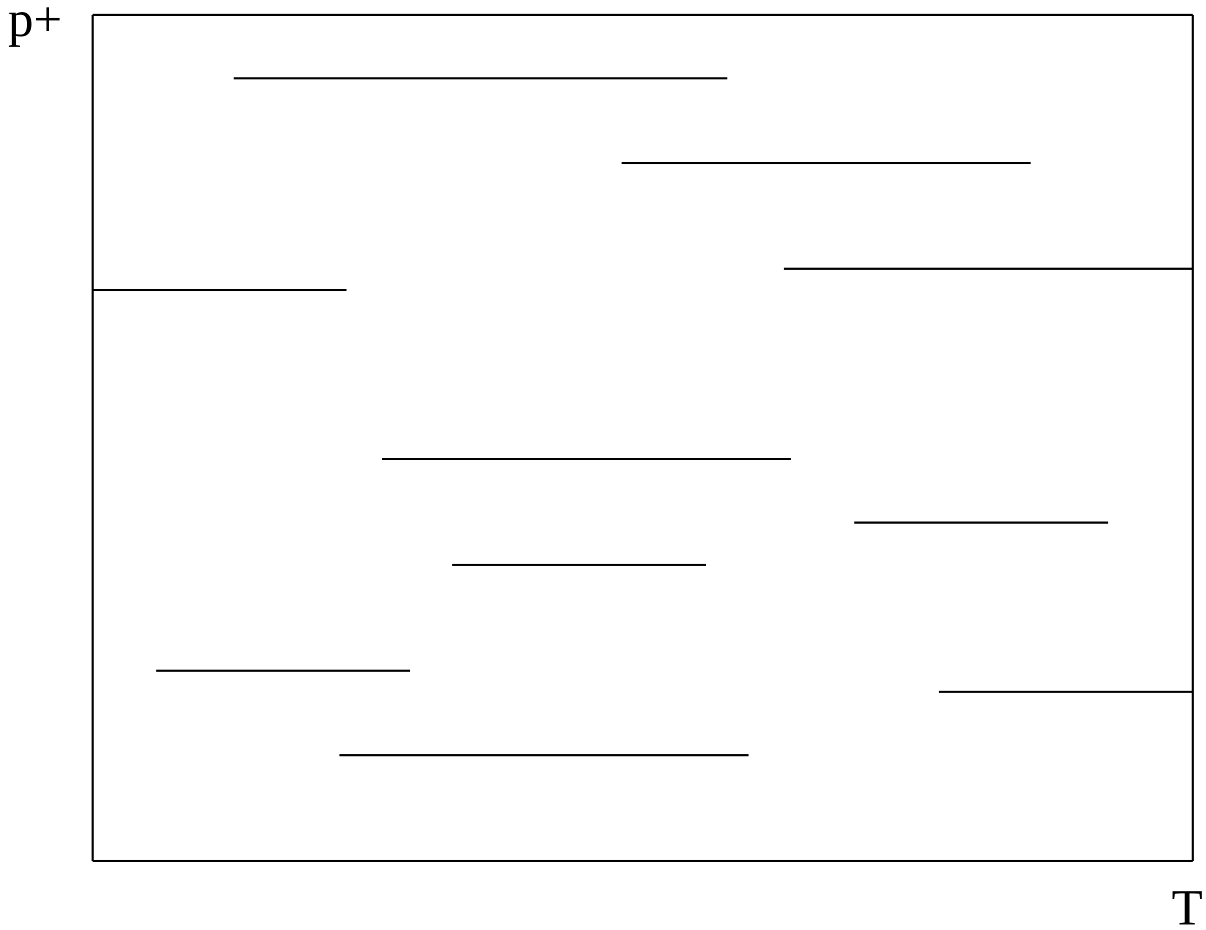
Properties of Protostring

- $d = 0$, $s = 24$ gives finite continuum limit w/o insertions.
- String in 1 space and 1 time dimension (x^+, x^-)
- This is a pure Grassmann analog of the bosonic string.
- Stable with no massless particles
- Degree of freedom count matches the superstring:

Bosonize 16 Grassmann fields $\Longrightarrow$ 8 (compactified) scalar fields.

- Superstring a deformation of this proto-string?

Scattering Amplitudes from Mandelstam



Open bosonic string string trees

$$\mathcal{A} = g^{N-2} \prod_{k=1}^{N} \frac{1}{|\alpha_k|^{d/48}} \int dZ_2 \cdots dZ_{N-2} \prod_{k<l} |Z_k - Z_l|^{2p_k \cdot p_l}$$
$$\left[\frac{1}{|\alpha_N|^{N-3}} \frac{\prod_{k<l} |Z_l - Z_k|}{\prod_{r<s} |x_s - x_r|} \right]^{(d-24)/24}$$

$$\rho = \tau + i\sigma = \sum_k \alpha_k \ln(z - Z_k), \qquad \left. \frac{d\rho}{dz} \right|_{z=x_r} = 0$$

x_r are the roots of a polynomial of order $N-2$!

For bosonic and super string x_r dependence disappears in critical dimension. Not so for the protostring!

Bosonization of Protostring, For $M \to \infty$,

$$\frac{S_k}{\sqrt{m}} \sim \frac{f_0}{\sqrt{P^+}} + \sqrt{\frac{2}{P^+}} \sum_{n=1}^{\infty} \left(f_n e^{-2i\pi n(T_0 t - \sigma)/P^+} + f_n^\dagger e^{+2i\pi n(T_0 t - \sigma)/P^+} \right)$$

$$\frac{\tilde{S}_k}{\sqrt{m}} \sim \frac{\tilde{f}_0}{\sqrt{P^+}} + \sqrt{\frac{2}{P^+}} \sum_{n=1}^{\infty} \left(\tilde{f}_n e^{-2i\pi n(T_0 t + \sigma)/P^+} + \tilde{f}_n^\dagger e^{+2i\pi n(T_0 t + \sigma)/P^+} \right)$$

$$f_0^2 = \tilde{f}_0^2 = 1, \qquad \{f_n, \tilde{f}_m\} = 0, \qquad \{f_n, f_m^\dagger\} = \{\tilde{f}_n, \tilde{f}_m^\dagger\} = \delta_{mn}$$

For a pair of Grassmann variables $f_n^{1,2}$ (and also for $\tilde{f}$):

$$a_n = \frac{i f_0^1}{\sqrt{2}} f_n^2 + f_n^1 \frac{i f_0^2}{\sqrt{2}} + i \sum_{k=1}^{n-1} f_k^1 f_{n-k}^2 + i \sum_{k=1}^{\infty} \left(f_k^{1\dagger} f_{n+k}^2 + f_{n+k}^1 f_k^{2\dagger} \right), \quad n > 0$$

$$a_0 = \frac{i}{2} f_0^1 f_0^2 + i \sum_{k=1}^{\infty} \left(f_k^{1\dagger} f_k^2 - f_k^{2\dagger} f_k^1 \right)$$

$$a_{-n} \equiv a_n^\dagger, \qquad [a_n, a_m] = n\delta_{n,-m}$$

Call a_0 or $\tilde{a}_0$ "helicity": its eigenvalues are $n+1/2$, n an integer.

$a_0 + \tilde{a}_0$, KK momentum, and $a_0 - \tilde{a}_0$, winding number have opposite parity. Assume $a_0 = \tilde{a}_0$ (0 winding) in following

Then can apply Mandelstam's bosonic string formalism, with momenta restricted to odd multiples of a fixed number γ.

Since momenta are quantized in half odd integers and since the three string vertex is non zero, the helicity is not conserved.

A nonzero three vertex requires insertion of $e^{\pm i\gamma\phi}$ at each break/join point. We propose hermitian insertion $\cos\gamma\phi$.

Notation for multi-string scattering

Uncompactified transverse d-momenta: $\boldsymbol{p}_k$

Compactified $(s/2)$-momenta (helicity): $\boldsymbol{\pi}_k$
Components of $\boldsymbol{\pi}_k$ are $(2n+1)\gamma$, where $\gamma = 1/(2\sqrt{2})$ for open strings.

$(s/2)$-momenta inserted at interaction point $\boldsymbol{\gamma}$
With all components $\pm\gamma$

Momentum/Helicity Conservation

$$\sum_{k=1}^{N} \boldsymbol{\pi}_k + \sum_{r=1}^{N-2} \boldsymbol{\gamma}_r = 0, \qquad \sum_{k=1}^{N} \boldsymbol{p}_k = 0$$

$\boldsymbol{P}_k \equiv (\boldsymbol{p}_k, \boldsymbol{\pi}_k)$, $P_k \equiv (p_k^-, p_k^+ . \boldsymbol{P}_k)$, and $p_k \equiv (p_k^-, p_k^+, \boldsymbol{p}_k)$.

Consider a generalized protostring with d bosonic and s Grassmann dimensions.

A finite continuum limit or $O(1,1)$ invariance fixes $d+s=24$ or $d=24-s$. Total number of boson fields $n_b = 24 - s/2$.

$$\left[I_O \left|\frac{\partial T}{\partial Z}\right| \det{}^{-(24-s/2)/2}(-\nabla^2)\right]_{\text{open}}$$

$$= \prod_{k=1}^{N} \frac{1}{\sqrt{|\alpha_k|}} \left[\frac{\prod_{k<N}|\alpha_k|}{|\alpha_N|}\right]^{s(1-8\gamma^2)/32} \left[\frac{\prod_{r<t}|x_t-x_r|\prod_{m<l}|Z_l-Z_m|}{\prod_{l,r}|Z_l-x_r|}\right]^{s/48}$$

$$\left[\frac{\prod_{r<s}|x_r-x_s|^{2\boldsymbol{\gamma}_r\cdot\boldsymbol{\gamma}_s-s\gamma^2/2}\prod_{k<l<N}|Z_k-Z_l|^{2P_k\cdot P_l-s\gamma^2/2}}{\prod_{r,k<N}|x_r-Z_k|^{-2\boldsymbol{\pi}_k\boldsymbol{\gamma}_r-s\gamma^2/2}}\right]$$

We see that the the three vertex scales properly if $\gamma^2 = 1/8$ in accord with the bosonization result. (If we had set $\gamma^2 = 1/8$ from the beginning this condition would be $d+s=24$.)

N **string scattering amplitude**:

$$\mathcal{A}_{\mathcal{N}} \;=\; \int dZ_2 \cdots dZ_{N-2}$$

$$\prod_{k=1}^{N} \frac{1}{\sqrt{|\alpha_k|}} \left[\frac{\prod_{r<s} |x_r - x_s|^{2\boldsymbol{\gamma}_r \cdot \boldsymbol{\gamma}_s - s/24} \prod_{k<l<N} |Z_k - Z_l|^{2P_k \cdot P_l - s/24}}{\prod_{r,k<N} |x_r - Z_k|^{-2\boldsymbol{\pi}_k \boldsymbol{\gamma}_r - s/24}} \right]$$

Maximal helicity violation and minimal mass: All γ_r the same; one $\pi_l = \gamma_r$ and $\pi_k = -\gamma_r$ for $k \neq l$.

Minimal mass 4 string amplitude
Maximal helicity violation

$$\boldsymbol{\gamma}_1 = \boldsymbol{\gamma}_2 = \boldsymbol{\pi}_4 = -\boldsymbol{\pi}_1 = -\boldsymbol{\pi}_2 = -\boldsymbol{\pi}_3$$

Then amplitude reads:

$$\begin{aligned} A_4^{\mathrm{open}} &= g^2 \prod_{k=1}^{4} \frac{1}{\sqrt{|\alpha_k|}} \int_0^1 dZ \\ & \quad \frac{|\alpha_4|^{s/4} |x_2 - x_1|^{s/12}}{|\alpha_1 \alpha_2 \alpha_3|^{s/12}} Z^{p_{12}^2 - 2 + s/12} (1-Z)^{p_{23}^2 - 2 + s/12} \end{aligned}$$

$$\alpha_4^2 |x_+ - x_-|^2 = \alpha_{12}^2 (1-Z) + \alpha_{23}^2 Z - \alpha_{13}^2 Z(1-Z)$$

where $\alpha_{kl} \equiv \alpha_k + \alpha_l$.

Protostring

Set $s = 24$, and by 1+1 kinematics either $\alpha_{23} = 0$ or $\alpha_{13} = 0$. In first case $\alpha_1/\alpha_2 = (S - 2 + \sqrt{S(S-4)})/2$:

$$\mathcal{A}^{\alpha_{23}=0} = \frac{g^2}{\alpha_1\alpha_2}(S - 2 + \sqrt{S(S-4)})^3 \frac{2}{(1-S)(3-S)}$$

where $S = -p_{12}^2$ and $p_{23}^2 = 0$. This is forward scattering. Backward scattering in this kinematics seems to be zero.

High energy behavior: $\mathcal{A} \propto$ constant.
As though a Regge trajectory with unit intercept.

Conclusion

I have tried to explain how to calculate scattering amplitudes for the protostring based on bosonization of Grassmann worldsheet firlds..

The protostring is an intersting toy model for how the string bit idea can give rise to string theory.

It is intriguingly close to superstring theory and we can hope it will lead to a better understanding of the physics of string.

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