Bethe-Salpeter Eq. Vector Meson-Scalar Meson Bounds States

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0.1 Bethe-Salpeter equation

In 1971, independently, the two-body relativistic, quantum field theory equation was independently obtained by H.. A. Bethe and E.E. Salpeter[1] and Julain Schwinger[2]. Although the derivations were quite different the equation obtained were equivalent. Its use in assessing the agreement between quantum electrodynamics cannot be over stated. It allowed the test of QED on electrons, muons etc. particles lacking internal structure that could obfuscate the results. The predictions from the the work of C. Sommerfled [3] , T. Fulton, D. Owen and W. Repko[4] based on the Bethe-Sapeter Equation have been borne out by experimental results.

The Bethe-Salper equation is the equation for the two-particle propagator. The condition for a the bound state to exist is that no particles are propagated to infinity. To be more specific, the two-particle quantum field theory propagator can be written as;

$$G(34;12) = -i < 0|T\{\psi_a^H(3)\psi_b^H(4)\bar{\psi}_a^H(1)\bar{\psi}_b^H(2)\}0 > \tag{1}$$

where the superscript H indicate that the fields are in the Heisenberg picture. No known exact solution are known for the the Bethe-Salpeter Eq. for physical systems but results can be obtained via perturbation theory. This involves writing Eq. 1 in the interaction picture where we obtain

$$\begin{aligned} G(34;12) &\equiv < 0|T\{\psi_a^H(3)\psi_b^H(4)\bar{\psi}_a^H(1)\bar{\psi}_b^H(2)\}|0 > \\ &= < 0|T\{\psi_a(3)\psi_b(4)\bar{\psi}_a(1)\bar{\psi}_b(2)S\}0 > \end{aligned} \tag{2}$$

with the S-matrix given by

$$\begin{aligned} S &= 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_n T(H_I(t_1) \cdots H_I(t_n)) \\ &\equiv T\left(\exp\left[-i\int_{-\infty}^{\infty} H_I(t)dt\right]\right) \end{aligned} \tag{3}$$

where the exponential form expressed in Eq 3 is a symbolic summary of the time ordered series with which it coincides when expanded in a power series in the interaction where the complete Hamiltonian can be written as H_0, containing the free particle Hamiltonian itself and H_I, the part containing interaction between the particles.

Bethe and Salpeter (as well as Schwinger) showed that Eq.8 satisfies the following equation which is commonly referred to as the Bethe-Salpeter Equation:

$$G(34;12) = S'_a(x_3 - x_1)_F S'_b(x_4 - x_2)_F + S'_a(x_3 - x_5)_F S'_b(x_4 - x_2)_F I(5,6;7,8)G(78;12) \tag{4}$$

where I is the irreducible part of H_i which connects particles 'a' and 'b' while the part which acts only on 'a' or 'b' iis called the reducible part and is the part which 'fully' dresses the propagators. Thus $S_a \to S'_a$ and $S_b \to S'_b$
To obtain an equation for a bound state from Eq.8 we need to insert a complete set of bound states $\sum_n = |n><n|$ into Eq. 8. For a bound state the inhomogenous term vanishes (otherwise the system would not be confined, thus not bound) and restricting ourselves to x_3^0 and $x_4^0 > x_1^0$ and x_2^0 and we define the two-particle wavefunction by $\phi_n(3,4) =< 0|T(\psi(3)\psi(4))|n >$,

The Bethe-Salpeter equation can now be written as

$$\phi_n(x_3, x_4) = S'_a(x_3 - x_5)S'_b(x_4 - x_6)I(x_5x_6; x_7x_8)\phi_n(x_7, x_8) \tag{5}$$

If we include the self-energy parts of each of the propagators as multiplicative factors of I, Eq. 5 can be written as

$$\phi_n(x_3, x_4) = S_a(x_3 - x_5)S_b(x_4 - x_6)\tilde{I}(56; 78)\phi_n(x_7, x_8) \tag{6}$$

where the propagators appearing in Eq. 6 are the free propagtors and $\tilde{I}$ is the irreducible kernel multiplied by the self-energies of each of the particles. In C.M. coordinates,

$$S_a(x_3 - x_5) = \int d^4p_1 e^{ip_1\cdot(x_3-x_5)} S_a(p_1) \tag{7}$$

etc. where $p_1 = \eta_1 K + p, p_2, p_2 = \eta_2 K - p, \eta_1 + \eta_2 = 1$ From going to the case, originally considered Bethe, Salpeter and Schwinger in whidh both of the bound partiles are fermions, must be modified when particles of integer spin are the considered. To be aware of the these probem, we discuss first the bound state in which both particles are fermions.

0.2 Fermon-fermion case-qed

$$\begin{aligned}
G(34;12) &\equiv < 0|T\{\psi_a^H(3)\psi_b^H(4)\bar{\psi}_a^H(1)\bar{\psi}_b^H(2)\}0 > \\
&= < 0|T\{\psi_a(3)\psi_b(4)\bar{\psi}_a(1)\bar{\psi}_b(2)S\}0 > \\
&= < 0|T\{\psi_a(3)\psi_b(4)\bar{\psi}_a(1)\bar{\psi}_b(2)\}0 > \\
&\quad + \sum_{n=1}^{\infty} < 0|T\{\psi_a(3)\psi_b(4)\bar{\psi}_a(1)\bar{\psi}_b(2) \\
&\quad \times \frac{(-i)^n}{n!}\int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_n T(H_I(t_1)\cdots H_I(t_n))\}0 >
\end{aligned} \tag{8}$$

where H is given by

$$H = H_0 + H_I \tag{9}$$

with $H_0 = i\gamma^a\partial + m_a + i\gamma^b\partial + m_b$

$$S_a(x_3 - x_5)S_b(x_4 - x_5) = \int e^{(X-X')\cdot K} e^{-i(p_1-p_2)} \frac{1}{[\gamma^a \cdot \eta_1 K + p - m_a][\gamma^b \cdot \eta_2 K - p - m_b]} \quad (10)$$

Separating rhs by partial fractions gives [5]

$$\frac{1}{[\gamma^a \cdot \eta_1 K + p - m_a][\gamma^b \cdot \eta_2 K - p - m_b]} = \left(\frac{1}{[\gamma^a \cdot \eta_1 K + p - m_a]} + \frac{1}{[\gamma^b \cdot \eta_2 K - p - m_b]}\right) \times \frac{1}{[K - H^a(\mathbf{p}) - H^b(\mathbf{-p})]} \quad (11)$$

where K is the energy eigenvalue. Equation 6 can be written as

$$[K - H^a(\mathbf{p}) - H^b(\mathbf{-p}) - \tilde{I}_0(x)]\phi_n(x) = 0 \quad (12)$$

$\tilde{I}$ is the irreducible kernel which is combined with first term on the right side of Eq. 11.

This was the initial form the Bethe-Salpeter equation and its perturbation theory was developed. It relied on the fact that the reciprocal of each of the free particles propagators, is linear in momentum.

0.3 Generalization Needed to Accomondate Vector Mesons as Bound State Constituents

It is the Eq.12 that connects quantum field treatment of bound states with both the non-relativistivic treatment as well as the external field approximaion using the Dirac equation. This above treatment relied on the inverse of the fermion propagators being linear in momentum. We want our formalism to describe bound states having constituents of scalar particles as well as charged vector mesons. In the standard form the inverse of these propagators are quadratic and not linear in the momentum. To have such particles described as constituents in the Bethe-Salpeter equation, in need to find an alternative form to express the propagators in which the momentum appears linear in the inverse propagator. In this way, we will have generalized the Bethe-Salpeter equation that the constituent particles could, in fact, be any of the particles occurring in the standard model. It is also conceivable that this formalism describe yet unobserved bound states whose constituents are predicted by supersymmetry. Up to this time, applications of the Bethe-Salpeter equation had been confined to interacting fermions via qed or quarks interacting with gluons. It needn't be so.

0.3.1 Bethe-Salpeter Equation with Spinless Particle Constituents

[6][7]

Two-Component Representation of the Klein-Gordon Propagator

The Klein-Gordon propagator can be defined as

$$\Delta_F(x-x') = -i < 0|T[\phi(x)\phi(x')]|0 > \tag{13}$$

where ϕ is the solution of the Klein-Gordon equation [i.e. $(\partial_x^2 + m^2)\phi(x) = 0$]. In momentum space Eq.13 can be written

$$\begin{aligned}\Delta_F(x-x') &= \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-x')} \frac{1}{p^2 - m^2 + i\epsilon} \\ &= \int \frac{d^3p}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} e^{-ip\cdot(x-x')} \frac{1}{2E_p} \left(\frac{1}{(p_0 - E_p - i\epsilon)} - \frac{1}{(p_0 + E_p - i\epsilon)} \right)\end{aligned} \tag{14}$$

where in Eq. 14 we have made a partial fraction decomposition of the propagator ($E_p = \sqrt{\mathbf{p}^2 + m^2}$). This propagator takes a free scalar particle state at x' and tranforms it to x, i.e.

$$\phi(x) = \int d^4x' \Delta_F(x-x')\phi(x') \tag{15}$$

It is easy to verify that if $\phi(x')$ is a positive energy eigenstate, then the second term in Eq.14 acting on ϕ vanishes and likewise, if ϕ is a negative energy state, then the first term of Eq.14 acting on it vanishes. Thus we can describe the action of Eq.15 as acting on a two-component wave function

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^- \end{pmatrix} \tag{16}$$

where the upper component represents a positive energy eigenstate and the lower component, a negative eigenstate. We can now write Eq.14 as

$$\Delta_F(x-x') = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-x')} \left(\frac{\Lambda^+}{2E_p(p_0 - E_p + i\epsilon)} - \frac{\Lambda^-}{2E_p(p_0 + E_p + i\epsilon)} \right) \tag{17}$$

where

$$\Lambda^{\pm} = \frac{1}{2}(1 + \pm\beta_s), \;\; \beta_s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{18}$$

In this two component representation, the quantum field representation of Φ, for a charged, spinless field is given by

$$\Phi(x) = \int \frac{d^3k}{\sqrt{2E_k(2\pi)^3}}\{ua_+(k)e^{-ik\cdot x} + va_-^\dagger(k)e^{ik\cdot x}\} \tag{19}$$

with $k_0 = E_k$,

$$u = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } v = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{20}$$

and $[a_\pm(k), a_\pm^\dagger(k')] = \delta(\mathbf{k} - \mathbf{k'}); [a_\pm(k), a_\pm(k')] = 0$, etc. with these definitions, one finds in a straightforward manner that $-i < 0|T[\Phi(x)\Phi(x')]|0 >$ leads directly to Eq.17. Furthermore, we can write the equation defining $\Delta_F(x - x')$ in this representation by

$$2E_p(p_0(\Lambda^+ - \Lambda^-) - E_p)\Delta_F(x - x') = \delta(x - x') \tag{21}$$

Alternatively, we coould write Eq.17 in a more compact form as

$$\Delta_F(x - x') = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-x')} \frac{1}{2p_0} \left(\frac{1}{p_0 - \beta_s E_p + i\epsilon} \right) \tag{22}$$

In what follows this form will be very convenient to use. It should, however, be noted that despite the appearance of a pole at $p_0 = 0$ in Eq.22 none actually occurs and the expression is finite as it must be at $p_0 = 0$ as it must be. This can be seen by writing the integrand of Eq.22 in a form similar to Eq.17. The form of Eq.22 is somewhat reminiscent of the Dirac propagator and correspondingly,leads to a simpler equation for $\Delta_F(x - x')$ than Eq.21. That is,

$$2p_0(p_0 - \beta_s E_p)\Delta_F(x - x') = \delta(x - x') \tag{23}$$

0.3.2 Bethe-Salpeter Equation in which one of the Constituents is a Vector Meson

Our example will be taken from the electro-weak theory. With the gauge fixing term [8]

$$\mathcal{L}_{gf} = -\frac{1}{2\xi}(\partial_\mu A^\mu - \xi M\phi_2)^2 \tag{24}$$

where ξ is a parameter determing the gauge. Depending on the value of ξ we get a class of gauges called t' Hooft gauges. Adding Eq.24 to the electro-weak Lagrangian we have

$$\mathcal{L} = \quad -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{M^2}{2}A^\mu A_\mu - \frac{1}{2\xi}\partial_\mu A^\mu \partial_\nu A^\nu + \frac{1}{2}(\partial_\mu\phi_1)^2$$
$$-\frac{1}{2}\lambda a^2\phi_1^2 + \frac{1}{2}(\partial_\mu\phi_2)^2 - \frac{\xi}{2}M^2\phi_2^2 + \text{coupling terms} \tag{25}$$

The quadratic term in A^μ is

$$\frac{1}{2}A^\mu[g_{\mu\nu}(\Box + M^2) - \partial_\mu\partial_\nu(1 - \frac{1}{\xi})]A^\nu \tag{26}$$

The vector propagator follows

$$D_{\mu\nu} = \frac{1}{k^2 - M^2}\left[-g_{\mu\nu} + (1-\xi)\frac{k_\mu k_{nu}}{k^2 - \xi M^2}\right] \tag{27}$$

If we choose the gauge $\xi \to \infty$, we obtain

$$D_{\mu\nu} = \frac{1}{k^2 - M^2}\left(-g_{\mu\nu} + \frac{k^\mu k^\nu}{M^2}\right) \tag{28}$$

From Eq.22 we know that

$$\frac{1}{k^2 - M^2 + i\epsilon} \to \frac{1}{2k_0}\left(\frac{1}{k_0 - \beta_s E_k + i\epsilon}\right) \tag{29}$$

Hence we can write the vector meson propagator by [9]

$$D_{\mu\nu} = \frac{1}{2k_0}\left(\frac{1}{k_0 - \beta_s E_k}\right)\left(-g_{\mu\nu} + \frac{k^\mu k^\nu}{M^2}\right) \tag{30}$$

where $E_k = \sqrt{k^2 + M^2}$

and where M is the mass of the vector meson.

Bethe-Salpeter Equation for a Vector-Meson and a Scalar Meson

As an example , we write the Bethe-Salpeter equation for a scalar meson and a vector meson particle.

$$[K - \beta_s E_a(k) - \beta_s E_b(k)]\, \phi_K^{\mu}(x) = \Lambda_K(x,x')\gamma_b^0 \left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{m_b^2} \right) I_K(x',x'')\phi_{K.\nu}(x'') \tag{31}$$

For electro-weak, the lowest order interaction Lagrangian can initially be used in H_{int} to obtain the first non-trivial equation, Fr this, there are two contributions: a) the scalar part interacting with the electromagnetic field, $H_{em,H}$ and b) the vector particle interaction with the electromagnetic field $H_{em,W}$

$$\mathcal{L}_{em,H} = ieA^{\lambda}\{(\partial_\lambda \bar{H})\frac{M}{2m_H}H - \bar{H}\frac{M}{2m_H}\partial_\lambda H\} - eA_0\bar{H}H + \text{higher-order terms} \tag{32}$$

$$\begin{aligned} \mathcal{L}_{em,W} \quad &= -ieA_\mu\{W_{nu}(\partial^\mu W^{*\mu} - \partial^\nu W^{*\nu}) - c.c.\} - i\frac{e}{2}F^{\mu\nu}\{W_\mu^* W_\nu - W^*{}_\mu W_\mu\} \\ &+e^2 A_\mu A_\nu[W^* W^\nu + \cdots] \end{aligned} \tag{33}$$

Using Eq. 32 and Eq. 33 we can now write

$$\begin{aligned} \mathcal{H}_{int} \quad = \quad & -ieA^{\lambda}\{(\partial_\lambda \bar{H})\frac{M}{2m_H}H - \bar{H}\frac{M}{2m_H}\partial_\lambda H\} + eA_0\bar{H}H + ieA_\mu\{W_{nu}(\partial^\mu W^{*\mu} - \partial^\nu W^{*\nu}) - c.c.\} \\ & +i\frac{e}{2}F^{\mu\nu}\{W_\mu^* W_\nu - W^*{}_\mu W_\mu\} \quad - e^2 A_\mu A_\nu[W^* W^\nu + \cdots] \end{aligned} \tag{34}$$

In our expansion of S the lowest order contributing terms are :

$$S = e^{\int \mathcal{H}_{int} dx} \approx 1 + \int d^4x_1 \mathcal{H}_{int} \int d^4x_2 \mathcal{H}_{int} \tag{35}$$

and from this, the interaction kernel I is readily calculated. Instead of proceeding on this track, the relevance of this calculation should be addressed which we do in the following section:

0.3.3 Bound State?

How do we decide if two interacting particles form a bound state when one of the particles is unstable? It would seem that if the 'bound system' can live long enough to revolve around each other (in a classical sense) many times, they could be considered bound. For example, for a system of a K^+-meson and an electron. The life-time of the K^+-meson is $\tau_{K+} \approx 1.2x10^{-8}sec.$. The Bohr radius for this system is approximately $33x10^{-11}$ meters. Before the kaon decays, the electron would have cycled the kaon approximately 500 times. In this sense, the kaon-electron could be considered a well-defined bound state. If this is our criteria and we apply it to the W-vector meson-Higgs particle system where $\tau_W \approx 10^{-27}$seconds, a similar calculstion one can conclude that before the W vector meson decays, the Higgs particle could have made only 1/1000 of one orbit. Clearly such a system cannot be considered a bound state by any criteria.

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