ANOMALOUS DIMENSIONS
AND THE RENORMALIZABILITY OF THE FOUR-FERMION INTERACTION

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1 THE FOUR-FERMION CHALLENGE

With the Weinberg-Salam-Glashow theory of spontaneously broken gauge invariance one can make $V^2 = \bar{\psi}\gamma_\mu\psi\bar{\psi}\gamma^\mu\psi$ and $A^2 = \bar{\psi}\gamma_\mu\gamma^5\psi\bar{\psi}\gamma^\mu\gamma^5\psi$ renormalizable by introducing intermediate vector bosons.

But what about $S^2 = [\bar{\psi}\psi]^2$ and $P^2 = [\bar{\psi}i\gamma^5\psi]^2$, for which there is no analog treatment.

Not just an academic question since the paradigm for dynamical symmetry breaking, the Nambu-Jona-Lasinio model, viz.

$$I_{\rm NJL} = \int d^4x \left(\bar{\psi}\gamma^\mu i\partial_\mu\psi - \frac{g}{2}[\bar{\psi}\psi]^2 - \frac{g}{2}[\bar{\psi}i\gamma^5\psi]^2\right), \tag{1}$$

is based on a chirally-symmetric $g[S^2+P^2]$ action, and for its point-like coupled vertices one has to use a cut-off. **So is the paradigm just a purely mathematical model or can it be made physical?** To do this one would have to make it **renormalizable**. In addition we will need to couple the Nambu-Jona-Lasinio model to gravity in order to handle the vacuum energy density and the cosmological constant that it generates.

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2 CRITICAL SCALING AND ANOMALOUS DIMENSIONS

Since the short-distance behavior of the four-fermion theory is independent of the fermion mass, let us for the moment take the fermion to be massless and the vacuum to be $|\Omega_0\rangle$. If there is conformal invariance (renormalization group fixed point), the short-distance behavior of the $\langle\Omega_0|T(\bar{\psi}(x)\psi(x)\bar{\psi}(0)\psi(0))|\Omega_0\rangle$ two-point function is given by

$$\langle\Omega_0|T(\bar{\psi}(x)\psi(x)\bar{\psi}(0)\psi(0))|\Omega_0\rangle=\frac{1}{\mu^{2\gamma_\theta}(x^2)^{d_\theta}}, \tag{2}$$

where μ is an off-shell subtraction point, that is needed in a massless theory, and $\gamma_\theta=d_\theta-3$ is the anomalous dimension of $\theta=\bar{\psi}\psi$. Fourier transforming then gives

$$\Pi^0_{\rm S}(q^2)=-i\int\frac{d^4p}{(2\pi)^4}{\rm Tr}\Big[\left(\frac{(-p^2)}{\mu^2}\frac{(-(p+q)^2)}{\mu^2}\right)^{\frac{\gamma_\theta}{4}}\frac{1}{\not{p}}\left(\frac{(-p^2)}{\mu^2}\frac{(-(p+q)^2)}{\mu^2}\right)^{\frac{\gamma_\theta}{4}}\frac{1}{\not{p}+\not{q}}\Big]. \tag{3}$$

On making a Dyson-Wick contraction we obtain

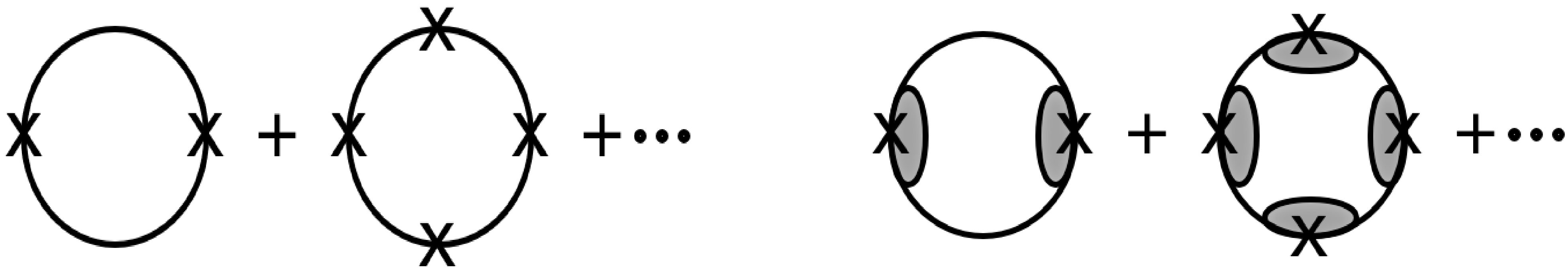
$$\Pi^0_{\rm S}(q^2)\;=\;-i\int\frac{d^4p}{(2\pi)^4}{\rm Tr}\Big[\tilde{S}_0(p)\tilde{\Gamma}^0_{\rm S}(p,p+q,q)\tilde{S}_0(p+q)\tilde{\Gamma}^0_{\rm S}(p+q,p,-q)\Big], \tag{4}$$

where $\tilde{S}_0(p)=1/\not{p}$, and $\tilde{\Gamma}^0_{\rm S}(p,p+q,q)$ is the Green's function associated with the insertion of the $\bar{\psi}\psi$ vertex with momentum q_μ into the inverse massless fermion propagator. On comparing we obtain

$$\tilde{\Gamma}^0_{\rm S}(p,p+q,q)\;=\;\left[\frac{(-p^2)}{\mu^2}\frac{(-(p+q)^2)}{\mu^2}\right]^{\frac{\gamma_\theta}{4}},\qquad \tilde{\Gamma}_{\rm S}(p,p,0)=\left(\frac{-p^2}{\mu^2}\right)^{\frac{\gamma_\theta}{2}}. \tag{5}$$

If $\gamma_\theta=0$ we obtain the point-coupled $\tilde{\Gamma}^0_{\rm S}(p,p+q,q)=1$, and $\Pi^0_{\rm S}(q^2)$ is **quadratically divergent**. Departures of γ_θ from zero thus provide for dressings of the Nambu-Jona-Lasinio point vertices.

3 RENORMALIZABILITY OF THE FOUR-FERMION INTERACTION TO LOWEST ORDER IN g



If we dress the $S^2 + P^2$ point vertices with $\tilde{\Gamma}^0_{\rm S}(p, p+q, q)$ vertices, the short-distance behavior of the theory will be softened if $\gamma_\theta < 0$, and will be softened so much if

$$\gamma_\theta = -1, \qquad d_\theta = 2 \tag{6}$$

that the $S^2 + P^2$ theory will potentially becoming renormalizable, with $\Pi^0_{\rm S}(q^2)$ becoming only logarithmically divergent, according to

$$\Pi^0_{\rm S}(q^2) = -\frac{\mu^2}{4\pi^2}\ln\left(\frac{\Lambda^2}{-q^2}\right). \tag{7}$$

In Mannheim 1975 it was suggested that $\gamma_\theta = -1$ gives renormalizability of the four-fermion interaction, and in Mannheim 2017 this was shown to be the case to all orders in g.

Since the massless theory has an unbroken chiral symmetry, we have

$$\Pi^0_{\rm P}(q^2)=-i\int\frac{d^4p}{(2\pi)^4}{\rm Tr}[\tilde{S}_0(p)\tilde{\Gamma}^0_{\rm S}(p,p+q,q)i\gamma^5\tilde{S}_0(p+q)\tilde{\Gamma}^0_{\rm S}(p+q,p,-q)i\gamma^5]=\Pi^0_{\rm S}(q^2). \quad (8)$$

To lowest order in g in the Bethe-Salpeter kernels for the scalar and the pseudoscalar channel fermion-antifermion scattering amplitudes $T^0_{\rm S}(q^2)$ and $T^0_{\rm P}(q^2)$, we obtain

$$\begin{aligned}T^0_{\rm S}(q^2) &= g+g\Pi^0_{\rm S}(q^2)g+....=\frac{1}{g^{-1}-\Pi^0_{\rm S}(q^2)},\\ T^0_{\rm P}(q^2) &= g+g\Pi^0_{\rm P}(q^2)g+....=\frac{1}{g^{-1}-\Pi^0_{\rm P}(q^2)}.\end{aligned} \quad (9)$$

In the massless fermion case we can thus choose g^{-1} to be a single **logarithmic divergence**, viz.

$$-\frac{\mu^2}{4\pi^2}\ln\left(\frac{\Lambda^2}{\mu^2}\right)=\frac{1}{g} \quad (10)$$

with $T^0_{\rm S}(q^2)$ and $T^0_{\rm P}(q^2)$ then both being finite, as per

$$T^0_{\rm S}(q^2)=T^0_{\rm P}(q^2)=\frac{4\pi^2}{\mu^2}\frac{1}{\ln\left[\mu^2/(-q^2)\right]}. \quad (11)$$

However, not only are $T^0_{\rm S}(q^2)$ and $T^0_{\rm P}(q^2)$ finite, they both have tachyons at $q^2=-\mu^2$ (degenerate since chiral symmetry unbroken), to thereby render the massless vacuum unstable. **The massless theory thus undergoes dynamical symmetry breaking**.

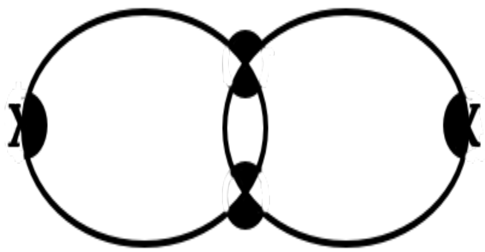
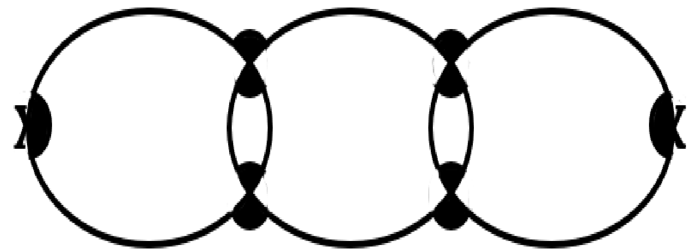
Specifically, what happens is that softening the theory in the ultraviolet makes it more divergent in the infrared. And at $\gamma_{\theta}=-1$ the infrared divergences become so severe that the chiral vacuum undergoes dynamical chiral symmetry breaking, with mass generation and long range order being produced (Mannheim 1974). Moreover, this breaking automatically generates a $1/g$ of precisely the needed log divergent form, viz. the massive fermion gap equation

$$-\frac{\mu^2}{4\pi^2}\ln\left(\frac{\Lambda^2}{M\mu}\right)=\frac{1}{g}, \tag{12}$$

where M is the dynamically generated fermion mass, to not only make the S^2+P^2 interaction be renormalizable in one loop, it automatically makes it finite in one loop too (Mannheim 2017). Comparing the massless and massive fermion expressions for $1/g$, we can set $M=\mu$.

But first, to show all-order renormalizability we need to consider graphs to all orders in g in the kernels.

5 ALL ORDER RENORMALIZABILITY OF THE FOUR-FERMION INTERACTION

At $\gamma_\theta = -1$ the massless fermion theory g^2 contribution is given by the two-loop graph

$$\Pi^0_{\rm S}(q^2,2) = -i\int \frac{d^4p}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}\frac{d^4r}{(2\pi)^4}\frac{(-\mu^2)^3g^2X}{Y}, \qquad Y = [p^2(p+q)^2(p+q+k)^2(p+k)^2r^2(r+k)^2]^{3/2},$$
$$X = {\rm Tr}[\not{p}(\not{p}+\not{q})(\not{p}+\not{q}+\not{k})(\not{p}+\not{k})]{\rm Tr}[\not{r}(\not{r}+\not{k})]. \tag{13}$$

We find that $\Pi^0_{\rm S}(q^2,2)$ diverges as a **single logarithm** of the form

$$\Pi^0_{\rm S}(q^2,2) \sim \mu^6 g^2 {\rm ln}\left[\frac{\Lambda^2}{(-q^2}\right]. \tag{14}$$

The same pattern repeats for the three-loop g^4 contribution $\Pi^0_{\rm S}(q^2,4)$, with each two powers of g adding in two powers of μ^2, two momentum integrations, four powers of momentum in the numerator, and 12 powers of momentum in the denominator. Thus no matter how many additional terms we add in to the kernel, $\Pi^0_{\rm S}(q^2,{\rm all}) = \Pi^0_{\rm S}(q^2) + \Sigma^\infty_{n=1}\Pi^0_{\rm S}(q^2,2n)$ continues to diverge as a **single logarithm** (i.e. no $\ln^2$ etc.). Hence, with just one subtraction $\Pi^0_{\rm S}(q^2,{\rm all})$ and $\Pi^0_{\rm P}(q^2,{\rm all})$ become **ultraviolet finite**.

An all-order single logarithmic divergence is common in conformal invariant theories. For instance in QED at a fixed point, the all-order QED vacuum polarization $\Pi_{\mu\nu}(z) = \langle\Omega|T(j_\mu(z)j_\nu(0))|\Omega\rangle$ is given by

$$\Pi_{\mu\nu}(z) = f(\alpha)(\eta_{\mu\nu}\partial_\alpha\partial^\alpha - \partial_\mu\partial_\nu)z^{-4}, \qquad \Pi_{\mu\nu}(q^2) = if(\alpha)(\eta_{\mu\nu}q^2 - q_\mu q_\nu){\rm ln}[\Lambda^2/(-q^2)], \tag{15}$$

where $f(\alpha)$ is a dimensionless (Gell-Mann Low) function of α.

To explicitly implement the needed subtraction in the four-fermion case, we rewrite g as $g = G/\mu^2$ where G is dimensionless, and can thus set

$$\Pi^0_{\rm S}(q^2,{\rm all}) = -\frac{\mu^2}{4\pi^2}F(G)\ln\left[\frac{\Lambda^2}{(-q^2)}\right], \qquad T^0_{\rm S}(q^2) = \frac{4\pi^2}{\mu^2\left(4\pi^2G^{-1}+F(G)\ln[\Lambda^2/(-q^2)]\right)}, \tag{16}$$

where $F(G)$ is power series in G^2 with $F(0) = 1$. We then obtain

$$T^0_{\rm S}(q^2) = \frac{4\pi^2}{\mu^2F(G)\ln[\mu^2/(-q^2)]} \tag{17}$$

if we require G to obey

$$4\pi^2G^{-1} + F(G)\ln\left[\frac{\Lambda^2}{\mu^2}\right] = 0. \tag{18}$$

For the typical case of $F(G) = 1 + G^2$ this requirement yields a leading behavior for G of the form

$$G = -\frac{4\pi^2}{\ln[\Lambda^2/\mu^2]} + \frac{(4\pi^2)^3}{\ln^3[\Lambda^2/\mu^2]} \tag{19}$$

for large cut-off, and thus $F(G) \to 1$. We thus obtain

$$T^0_{\rm S}(q^2) = \frac{4\pi^2}{\mu^2\ln[\mu^2/(-q^2)]}, \qquad T^0_{\rm P}(q^2) = \frac{4\pi^2}{\mu^2\ln[\mu^2/(-q^2)]} \tag{20}$$

and the scattering amplitudes are finite.

The presence of two degenerate tachyons thus persists to all orders in g, and thus the vacuum breaking persists to all orders.

6 DYNAMICAL SYMMETRY BREAKING TO LOWEST ORDER IN g

We shall use the illustrative QED at a fixed point to generate the conformal invariance, since when $\beta(\alpha)=0$ the massive fermion QED

$$\left[m\frac{\partial}{m}+\beta(\alpha)\frac{\partial}{\partial\alpha}\right]\tilde{S}_m^{-1}(p)=-m[1-\gamma_\theta(\alpha)]\tilde{\Gamma}^m_{\rm S}(p,p,0) \tag{21}$$

has an asymptotic solution

$$\tilde{S}_m^{-1}(p)\ =\ \not{p}-m\left(\frac{-p^2}{m^2}\right)^{\frac{\gamma_\theta}{2}},\qquad \tilde{\Gamma}^m_{\rm S}(p,p,0)=\left(\frac{-p^2}{m^2}\right)^{\frac{\gamma_\theta}{2}}. \tag{22}$$

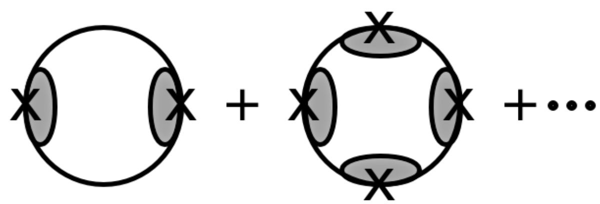
We take as chiral-symmetric action **massless** QED coupled to a four-fermion theory with action $I^0_{\rm QED}+I_{\rm FF}$, where

$$I^0_{\rm QED}=\int d^4x\Big[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\bar{\psi}\gamma^\mu(i\partial_\mu-eA_\mu)\psi\Big],\qquad I_{\rm FF}=\int d^4x\Big[-\frac{g}{2}[\bar{\psi}\psi]^2-\frac{g}{2}[\bar{\psi}i\gamma^5\psi]^2\Big], \tag{23}$$

and introduce a trial mass term in order to reexpress the action in mean field (MF) and residual interaction (RI) terms

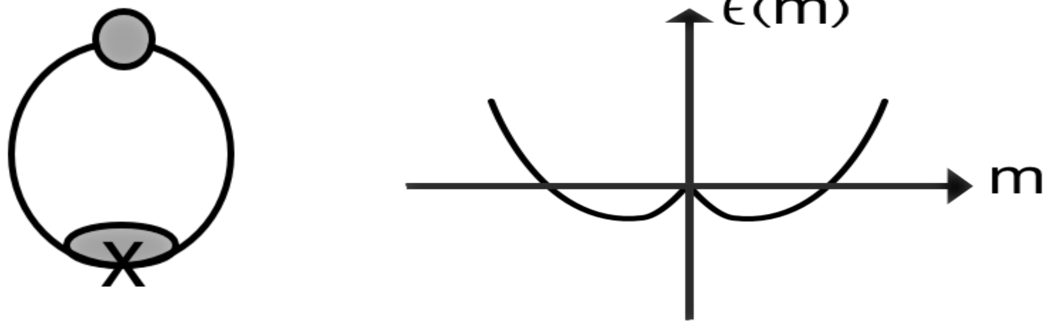
$$I_{\rm MF}\ =\ I^0_{\rm QED}+\int d^4x\left(-m\bar{\psi}\psi+\frac{m^2}{2g}\right),\qquad I_{\rm RI}=\int d^4x\left(-\frac{g}{2}\right)\left(\left(\bar{\psi}\psi-\frac{m}{g}\right)^2+\left(\bar{\psi}i\gamma^5\psi\right)^2\right). \tag{24}$$

When $\gamma_\theta=-1$ the mean-field vacuum energy density $\epsilon(m)$ is given by (Mannheim 1975)



$$\epsilon(m)=i\int\frac{d^4p}{(2\pi)^4}{\rm Tr}\left[\ \ln(\tilde{S}_\mu^{-1}(p,m))-\ln(\not{p})\right]=-\frac{m^2\mu^2}{8\pi^2}\left[\ln\left(\frac{\Lambda^2}{m\mu}\right)+\frac{1}{2}\right],\ \tilde{S}_\mu^{-1}(p,m)=\not{p}-m\tilde{\Gamma}^0_{\rm S}(p,p,0)=\not{p}-m\left(\frac{-p^2}{\mu^2}\right)^{\frac{\gamma_\theta}{2}}, \tag{25}$$

so that the massive vacuum lies lower than the massless vacuum. When $-1<\gamma_\theta<0$ $\epsilon(m)$ is a single well, while if $\gamma_\theta<1$ $\epsilon(m)$ is an upside down single well. Thus $\gamma_\theta=-1$ is singled out, and it is just we want for renormalizability.



By setting

$$\langle \Omega_M | \left(\bar{\psi}\psi - \frac{M}{g} \right)^2 | \Omega_M \rangle = \langle \Omega_M | \left(\bar{\psi}\psi - \frac{M}{g} \right) | \Omega_M \rangle^2 \ = \ 0 \tag{26}$$

in the mean-field, Hartree-Fock approximation, we define the physical vacuum to be that vacuum in which

$$\langle \Omega_M | H_{\rm RI} | \Omega_M \rangle = 0, \qquad \langle \Omega_M | \bar{\psi}\psi | \Omega_M \rangle = \frac{M}{g}. \tag{27}$$

And with $\langle \Omega_m | \bar{\psi}\psi | \Omega_m \rangle = \epsilon'(m)$ for any m, we obtain the **gap equation**

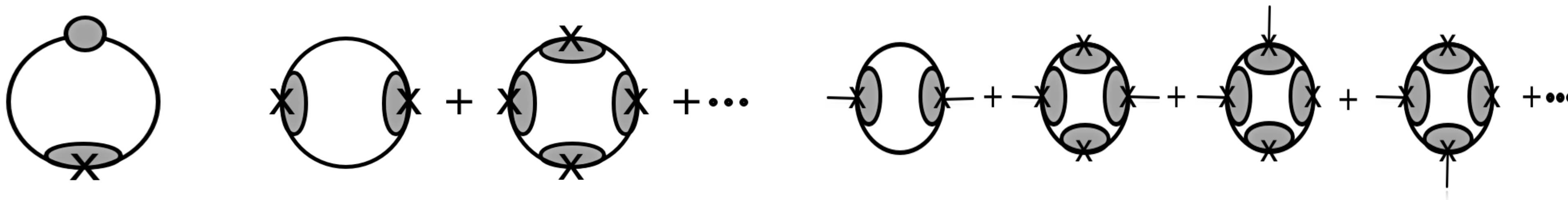
$$\langle \Omega_M | \bar{\psi}\psi | \Omega_M \rangle = -i \int \frac{d^4p}{(2\pi)^4} {\rm Tr}[\tilde{\Gamma}^0_{\rm S}(p,p,0)\tilde{S}_{\mu}(p,M)] = i \int \frac{d^4p}{4\pi^4} \frac{M\mu^2}{(p^2)^2 + M^2\mu^2} = -\frac{M\mu^2}{4\pi^2} \ln\left(\frac{\Lambda^2}{M\mu}\right) = \frac{M}{g},$$
$$M\mu = \Lambda^2 \exp\left(\frac{4\pi^2}{\mu^2 g}\right). \tag{28}$$

The gap equation has a non-trivial solution no matter how small g might be, as long as it is negative, viz. attractive as per its definition in $I_{\rm FF}$, with symmetry breaking being obtained for **weak coupling**, and with there being no need for the strong coupling that is widely thought to be required for dynamical symmetry breaking. We obtain the **essential singularity** behavior familiar from the BCS theory of superconductivity. And with there being an $m^2/2g$ term in $I_{\rm MF}$, we obtain

$$\tilde{\epsilon}(m) = \epsilon(m) - \frac{m^2}{2g} = \frac{m^2\mu^2}{16\pi^2} \left[\ln\left(\frac{m^2}{M^2}\right) - 1 \right], \tag{29}$$

with the mean-field induced cosmological $-m^2/2g$ term **automatically** making $\tilde{\epsilon}(m)$ be completely **finite** without the need for any fine tuning. We recognize $\tilde{\epsilon}(m)$ as having the shape of a **dynamically induced double-well potential**, with a minimum at $m = M$, and with $\tilde{\epsilon}(m)$ serving as a completely finite cosmological constant, one ultimately controlled when the four-fermion theory is coupled to the equally conformal invariant conformal gravity theory (Mannheim 2017).

7 DYNAMICAL SYMMETRY BREAKING TO ALL ORDERS IN g



We note that g^{-1}, $\Pi^0_{\rm S}(q^2)$ and thus $\Pi^m_{\rm S}(q^2)$ all have precisely the same dependence on Λ. Consequently, with this g^{-1}, $T^0_{\rm S}(q^2)$ and $T^0_{\rm P}(q^2)$, and thus $T^m_{\rm S}(q^2)$ and $T^m_{\rm P}(q^2)$, are all automatically finite. Symmetry breaking thus precisely provides the subtraction need to make both $T^m_{\rm S}(q^2)$ and $T^m_{\rm P}(q^2)$ be finite, **with the theory doing it automatically all on its own**.

The reason why we get an automatic cancellation of ultraviolet divergences is that in the expansions of $\epsilon(m)$, $\Pi^0_{\rm S}(q^2)$ and $\Pi^0_{\rm P}(q^2)$, the only ultraviolet divergent graphs are those with **two** $\bar{\psi}\psi$ insertions. **Thus $\epsilon(m)$, $\Pi^0_{\rm S}(q^2)$ and $\Pi^0_{\rm P}(q^2)$ all have the identical ultraviolet divergence structure, and not just in lowest order but even after being dressed to all orders in g.** The divergent part of $\epsilon(m)$ is given by $(1/2)G^{(2)}_0(q_\mu = 0, m = 0)m^2$, with the divergent part of $\epsilon'(m)$ thus being given by $G^{(2)}_0(q_\mu = 0, m = 0)m$. And with $G^{(2)}_0(q_\mu = 0, m = 0)$, $\Pi^0_{\rm S}(q_\mu = 0)$, and $\Pi^0_{\rm P}(q_\mu = 0)$ all being identically equal in the massless theory, on identifying $\epsilon'(m)$ with m/g, the cancellation automatically follows.

8 THE ROLE OF THE DYNAMICAL GOLDSTONE BOSON

To reinforce the point, we look for the Goldstone boson that must be present if there is to be dynamical mass generation since $I^0_{\rm QED} + I_{\rm FF}$ is chirally symmetric. The needed massive theory $\Pi^m_{\rm S}(q^2)$ and $\Pi^m_{\rm P}(q^2)$ are given by

$$\begin{aligned}\Pi^m_{\rm P}(q^2) &= -i\int\frac{d^4p}{(2\pi)^4}{\rm Tr}[\tilde{S}_\mu(p,m)\tilde{\Gamma}^0_{\rm S}(p,p+q,q)i\gamma^5\tilde{S}_\mu(p+q,m)\tilde{\Gamma}^0_{\rm S}(p+q,p,-q)i\gamma^5],\\ \Pi^m_{\rm S}(q^2) &= -i\int\frac{d^4p}{(2\pi)^4}{\rm Tr}[\tilde{S}_\mu(p,m)\tilde{\Gamma}^0_{\rm S}(p,p+q,q)\tilde{S}_\mu(p+q,m)\tilde{\Gamma}^0_{\rm S}(p+q,p,-q)],\end{aligned} \tag{30}$$

where $\tilde{S}^{-1}_\mu(p,m) = \not{p} - m\left(-p^2/\mu^2\right)^{\gamma_\theta/2}$. Similarly, the scattering amplitudes are given by

$$T^m_{\rm S}(q^2) = \frac{1}{g^{-1} - \Pi^m_{\rm S}(q^2)}, \qquad T^m_{\rm P}(q^2) = \frac{1}{g^{-1} - \Pi^m_{\rm P}(q^2)}. \tag{31}$$

When $\gamma_\theta = -1$ and $m = M$, and recalling the gap equation, we have

$$\Pi^M_{\rm P}(q^2=0) = i\int\frac{d^4p}{4\pi^4}\frac{\mu^2}{(p^2)^2 + M^2\mu^2}, \qquad i\int\frac{d^4p}{4\pi^4}\frac{M\mu^2}{(p^2)^2+M^2\mu^2} = -\frac{M}{g} \tag{32}$$

We see that there indeed is a massless pseudoscalar Goldstone boson pole in $T^M_{\rm P}(q^2)$, just as required. Moreover, since there has to be such a Goldstone pole if there is to be dynamical chiral symmetry breaking, the log divergences in $1/g$ and in $\Pi^M_{\rm S}(q^2)$ have no choice but to cancel each other identically.

Suppose we now go beyond Hartree-Fock and start dressing the loops with higher order $I_{\rm FF}$ interactions. The Goldstone pole must survive, and thus the cancellation of the log divergence must persist, with the all order in g dressings of $\epsilon'(M) = \langle \Omega_M|\bar{\psi}\psi|\Omega_M\rangle = M/g$ and of $T_{\rm P}^M(q^2=0)$ both automatically yielding the previously found condition

$$4\pi^2 G^{-1} = -F(G)\ln[\Lambda^2/M\mu]. \tag{33}$$

Thus a condition that was initially imposed in order to control the ultraviolet behavior of $I^0_{\rm QED}+I_{\rm FF}$ is now found to automatically emerge as a constraint provided by the consistency of the infrared structure of the theory.

ALL-ORDER FINITENESS OF THE FOUR-FERMION THEORY IS THUS ENFORCED BY THE GOLDSTONE THEOREM.

9 A DYNAMICAL HIGGS – CHIRAL PARTNER OF THE GOLDSTONE BOSON

Because of the chiral symmetry, in the same way that the massive $T_{\rm P}^M(q^2)$ contains a bound state Goldstone pole, the massive $T_{\rm S}^M(q^2)$ contains a dynamical scalar Higgs boson. Interestingly, explicit calculation (Mannheim 2017) shows that it actually lies above the threshold in the scalar channel fermion-antifermion scattering amplitude, to thus be **a resonance with a width**, rather than a bound state. In Mannheim 2017 it was suggested that **this width could serve as a diagnostic to distinguish between a dynamical Higgs boson and the elementary one that would appear in a fundamental Lagrangian**.

10 THE VACUUM ENERGY DENSITY PROBLEM AND THE POWER OF DYNAMICAL SYMMETRY BREAKING

Consider a free fermion of mass m in flat, four-dimensional spacetime. With $k^\mu = ((k^2 + m^2/\hbar^2)^{1/2}, \bar{k})$ evaluating the vacuum expectation value of the matter field energy-momentum tensor $T^{\rm m}_{\mu\nu} = i\hbar\bar{\psi}\gamma_\mu\partial_\nu\psi$ yields

$$\langle\Omega|T^{\rm m}_{00}|\Omega\rangle = i\int\frac{d^4p}{(2\pi)^4}{\rm Tr}\left[\,\ln(\not{p}-m)\right] = -\frac{\Lambda^4}{16\pi^2}\ln\Lambda^2 + \frac{\Lambda^4}{32\pi^2} - \frac{m^2\Lambda^2}{8\pi^2} + \frac{m^4}{16\pi^2}\ln\left(\frac{\Lambda^2}{m^2}\right) + \frac{m^4}{32\pi^2}, \qquad (34)$$

with Λ being a cut-off. We identify **QUARTIC, QUADRATIC AND LOGARITHMIC DIVERGENCES**. In flat space we can eliminate the quartic divergence by normal ordering, with the vacuum energy density then being given by the quadratically divergent

$$\epsilon(m) = i\int\frac{d^4p}{(2\pi)^4}{\rm Tr}\left[\ln(\not{p}-m) - \ln(\not{p})\right] = -\frac{m^2\Lambda^2}{8\pi^2} + \frac{m^4}{16\pi^2}\ln\left(\frac{\Lambda^2}{m^2}\right) + \frac{m^4}{32\pi^2}. \qquad (35)$$

When we dress the bare point vertices with $\gamma_\theta = -1$, $\epsilon(m)$ is then only the logarithmically divergent

$$\epsilon(m) = i\int\frac{d^4p}{(2\pi)^4}{\rm Tr}\left[\,\ln\left(\not{p} - m\left(\frac{-p^2}{\mu^2}\right)^{-1/2}\right) - \ln(\not{p})\right] = -\frac{m^2\mu^2}{8\pi^2}\left[\ln\left(\frac{\Lambda^2}{m\mu}\right) + \frac{1}{2}\right]. \qquad (36)$$

However, the mean-field generated a $-m^2/2g$ vacuum term for us automatically, with the mean-field vacuum energy density then being given by the completely finite

$$\tilde{\epsilon}(m) = \epsilon(m) - \frac{m^2}{2g} = \frac{m^2\mu^2}{16\pi^2}\left[\ln\left(\frac{m^2}{M^2}\right) - 1\right]. \qquad (37)$$

Hence with $\gamma_\theta = -1$ dynamical symmetry breaking not only automatically occurs, it automatically generates a vacuum $-m^2/2g$ term, to thus render $\tilde{\epsilon}(m)$ completely finite, doing so not only to lowest order in g but to all orders. Without dynamical symmetry breaking (i.e. in $|\Omega_0\rangle$) $\gamma_\theta = -1$ would render the four-fermion theory renormalizable, but with it (i.e. in $|\Omega_M\rangle$) the four-fermion theory is rendered completely finite. **This then is the power of dynamical symmetry breaking**.

11 BUT IN THE PRESENCE OF GRAVITY WE CANNOT NORMAL ORDER

Now as long as we ignore gravity we can only measure energy density differences, and thus it is okay to normal order the quartic divergence, as all one needs for dynamical symmetry breaking is to show that $\epsilon(m)$ is negative, i.e. that the massive fermion vacuum lies lower than the massless fermion vacuum, with $\epsilon(m)$ then being $\epsilon(m) = \langle\Omega_m|H_{\rm MF}|\Omega_m\rangle - \langle\Omega_0|H_{\rm MF}|\Omega_0\rangle$.

However, once one couples to gravity, **the hallmark of Einstein gravity is that gravity couples to energy density itself and not to energy density difference**. Thus in the presence of gravity we need to cancel the quartic divergence as well. Noting that in the mean-field approximation the vacuum energy density is a zero-point energy density, to cancel it we need another zero-point contribution.

However, to cancel the fermionic contribution we would need a contribution with the opposite sign, hence we would need a boson. If a theory of gravity can make sense quantum-mechanically, then the graviton zero-point energy density would nicely do the job. Since Einstein gravity is not renormalizable we cannot use Einstein gravity. However **conformal gravity**, viz. gravity based on the action

$$I_{\rm W} = \int d^4x(-g)^{1/2}C_{\lambda\mu\nu\tau}C^{\lambda\mu\nu\tau} \tag{38}$$

where $C_{\lambda\mu\nu\tau}$ is the Weyl conformal tensor) is immediately suggested since it too has an underlying conformal structure. The theory is renormalizable and ghost free and unitary (Bender and Mannheim 2008). And then (Mannheim 2017) one can not only cancel the quartic divergence, one can even control the cosmological constant.

12 SUMMARY

(1) To cancel the quartic divergence in the vacuum energy density need conformal gravity.

(2) With conformal invariance and $\gamma_\theta = -1$ one can soften the quadratic divergence in the vacuum energy density to make it only logarithmically divergent.

(3) With conformal invariance and $\gamma_\theta = -1$ dynamical symmetry breaking occurs, and an $-m^2/2g$ vacuum energy density term is induced, which then cancels the logarithmic divergence and makes the vacuum energy density completely finite.

(4) With conformal invariance and $\gamma_\theta = -1$, the $g[S^2 + P^2]$ four-fermion interaction gives finite scattering amplitudes to all orders in g.

(5) Conformal gravity coupled to a $g[S^2 + P^2]$ four-fermion interaction and a gauge theory that generates conformal invariance and $\gamma_\theta = -1$ can thus serve as a completely finite theory of everything, one in which the cosmological constant is completely under control and all masses are dynamical (Mannheim 2017).