Gravitational duality and deformations of action principles for mixed-symmetry tensor fields

Sergio Hörtner

Miami Conference 2017

Outline:

I. Some remarks on electric-magnetic duality

II. The Curtright action

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I.Some remarks on electric-magnetic duality

Maxwell equations in vacuum

$$\begin{array}{ll} \nabla \cdot \mathbf{E} = 0 & \nabla \cdot \mathbf{B} = 0 \\ \dfrac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} = 0 & \dfrac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0 \end{array}$$

are invariant under $SO(2)$ rotations acting on the internal space spanned by **E** and **B**

$$\begin{aligned} \mathbf{E} \quad &\to \mathbf{E}' = \cos\alpha \mathbf{E} + \sin\alpha \mathbf{B} \\ \mathbf{B} \quad &\to \mathbf{B}' = -\sin\alpha \mathbf{E} + \cos\alpha \mathbf{B} \end{aligned}$$

Interpretation: duality between electricity and magnetism (the existence of two alternative, physically equivalent formulations of Maxwell equations with the rôle of electric and magnetic degrees of freedom exchanged).

It can also be interpreted as a solution generating technique: given a solution $(\mathbf{E}(x), \mathbf{B}(x)) = (f(x), g(x))$ of Maxwell equations, one can construct different solutions $(\mathbf{E}'(x), \mathbf{B}'(x))$ by the action of this transformations.

Can duality be implemented as a symmetry in the action principle?

The answer is yes (Deser-Teitelboim, 1976). Key: use the Hamiltonian formalism and solve the Gauss law. This introduces a new 3d vector potential. After substitution in the action principle, a $SO(2)$ symmetry holds.

Duality is a "hidden symmetry".

The action principle in its Hamiltonian form is

$$S[A_i, \pi^j, A_0] = \int dt \; d^3x [\pi^i \dot{A}_i - \mathcal{H} - A_0 C]$$

with the Hamiltonian density

$$\mathcal{H} = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2)$$

and the constraint

$$C \equiv \partial^i E_i$$

A_0 is a Lagrange multiplier

Solving for the constraint and substituting in the action principle:

$$S[A^a{}_i] = \int d^4x(\epsilon_{ab}B^a\dot{A}^b - \delta_{ab}B^aB^b) \quad a,b = 1,2$$

where

$$B^a = \nabla \times A^a$$

Gauge symmetries:

$$\delta A^a{}_i = \partial_i v^a$$

This action principle is manifestly invariant under $SO(2)$ rotations

$$\begin{aligned} A^1 &\rightarrow A'^1 = \cos\alpha A^1 + \sin\alpha A^2 \\ A^2 &\rightarrow A'^2 = -\sin\alpha A^1 + \cos\alpha A^2 \end{aligned}$$

The result also holds in curved backgrounds.

Manifest duality invariance seems to require non-manifest Lorentz invariance.
Lorentz-invariance is hidden and can be checked by the fulfillment of Dirac-Schwinger commutator relations.
Recover Lorentz covariance?

Topology of space-time is assumed to be trivial.

A similar analysis can be performed for the Pauli-Fierz action:

$$S[h_{\mu\nu}] = -\frac{1}{4}\int d^4x[\partial^\rho h^{\mu\nu}\partial_\rho h_{\mu\nu} - 2\partial_\mu h^{\mu\nu}\partial^\rho h_{\rho\nu} + 2\partial^\mu h^\rho{}_\rho \partial^\nu h_{\mu\nu} - \partial^\mu h_\rho{}^\rho \partial_\mu h_\sigma{}^\sigma]$$

Linearized diffeomorphisms: $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$

A 3+1 decomposition and the introduction of the conjugate momentum yields

$$S[h_{ij}, \pi^{ij}, n, n_i] = \int dt \left(\int d^3x \, \pi^{ij} \dot{h}_{ij} - H - \int d^3x (n\mathcal{C} + n_i \mathcal{C}^i) \right)$$

Hamiltonian:

$$H = \int d^3 \left[\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2 + \frac{1}{4}\partial^m h^{ij}\partial_m h_{ij} - \frac{1}{2}\partial_m h^{mn}\partial_r h^r{}_n + \frac{1}{2}\partial^m h \partial^n h_{mn} - \frac{1}{4}\partial^m h \partial_m h \right]$$

Constraints:

$$\begin{aligned} \mathcal{C} &\equiv \partial^i \partial^j h_{ij} - \Delta h = 0 \\ \mathcal{C}^i &\equiv -2\partial_j \pi^{ij} = 0 \end{aligned}$$

The resolution of the constraints yields

$$\pi_{ij} = \epsilon^{imn}\epsilon^{jkl}\partial_m\partial_k P_{nl}$$
$$h_{ij} = \epsilon_{imn}\partial^m\phi_j{}^n + \epsilon_{jmn}\partial^m\phi_i{}^n + \partial_i u_j + \partial_j u_i$$

(Henneaux-Teitelboim, 2005)
ϕ_{ij} and P_{ij} are two symmetric potentials and u_i is a vector prepotential.

The gauge transformations acting on the potentials are

$$\delta Z^a_{ij} = \partial_i\eta^a_j + \partial_j\eta^a_i + \delta_{ij}\eta^a$$
$$(Z_\alpha{}^{ij}) = (P^{ij}, \phi^{ij}) \quad \alpha = 1, 2$$

These transformations have the form of the symmetries of conformal gravity.

The action can be written as (Bunster-Henneaux-Hörtner)

$$S[Z_\alpha{}^{ij}] = \int dt \left(-2\int d^3x \epsilon^{\alpha\beta} D^{ij}_\alpha \dot{Z}_{\beta\ ij} - \int d^3x (4R^\alpha_{ij} R^{\beta\ ij} - \frac{3}{2} R^\alpha R^\beta)\delta_{\alpha\beta} \right)$$

$D^{ij}[Z^a] = \epsilon^{iab} C_{ab}{}^j$ and $R_{ij}[Z^a]$ are respectively the dual of the Cotton tensor and the Ricci tensor constructed out of the prepotentials.

This action is manifestly invariant under the $SO(2)$ rotations.

Similar results hold in the case of lienarization of the ADM action around a de Sitter background using planar coordinates (Julia, Levie, Ray 2005).

Electric-magnetic duality appears in the study of the hidden symmetries of gravitational theories: Ehlers phenomenon.

Pure gravity in the presence of a (non-null) Killing vector exhibits a hidden $SL(2, R)$ symmetry.

Ehlers recognized a set of transformations relating to every known static solution of Einstein equations a whole family of physically non-equivalent stationary solutions. These transformations act on suitably defined potentials constructed out of the components of the metric. They define a $SO(2)$ symmetry group that can be thought of as interchanging "electric" and "magnetic" aspects of gravity. It can be extended to a $SL(2, R)$ symmetry group (inclusion of transformations that do not change the geometry of the solutions).

Einstein-Hilbert action (parametrized accordingly to the existence of a Killing vector) reduced to three dimensions exhibits a hidden $SL(2, R)$ symmetry after dualisation of Kaluza-Klein vector to a scalar. The scalar sector is described by a $SL(2, R)/SO(2)$ coset space.

In the presence of two commuting Killing vectors, infinite-dimensional symmetry group (Geroch group). Lie algebra structure: $sl(2, R)$ affine Kac-Moody algebra (Julia).

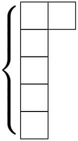
A similar phenomenon is encountered in 11d supergravity: 11d supergravity compactified on a (11-d)-torus yields maximally extended supergravity in dimension d with a hidden symmetry G/H governing the non-gravitational part of the reduced bosonic sector (p-forms and scalars). p-forms combine in an irreducible representation of a non-compact group G acting globally, whereas the scalar sector is described by the non-linear sigma model G/H with $H$the maximal compact subgroup of G.

In even dimensions, the global symmetry G is realized as a duality transformation interchanging Bianchi identities and equations of motion.

Reduction to five, four and three dimensions yields as the global, non-compact group G the exceptional Lie groups $E_{6(6)}$, $E_{7(7)}$, $E_{8(8)}$ respectively.

It has been conjectured that some indefinite Kac-Moody algebra (E_{10}, E_{11}) might be a fundamental symmetry of 11d supergravity/M-theory (Julia, Nicolai, Damour, Henneaux, West,...). These algebras treat fields and their Hodge duals on an equal footing, in particular the graviton and its dual field.

Dual graviton: a $(D-3,1)$ mixed-symmetry tensor. In five dimensions, it is described by the Curtright field: a (2,1) tensor.

$D-3$ boxes


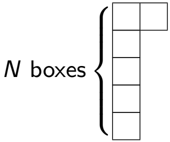
We expect to have action principles involving all these fields and their duals, the exceptional symmetry relating all of them in a highly non-trivial way.

The mixed-symmetry character of the dual graviton brings in difficulties in the construction fo action principles for E theory. Lack of a notion of diffeomorphism invariance for mixed-symmetry tensor fields. No-go result: no consistent deformations of the free theory for the Curtright field under the hypotheses of locality and manifest space-time invariance (Bekaert, Boulanger, Henneaux 2005).

II. The Curtright action

The study of gauge theories of mixed-symmetry tensor fields ("generalized gauge fields") was motivated by the emergence in string field theory of massive higher spin excitations transforming in arbitrary representations of the Lorentz group (mid-80's), including mixed-symmetry representations.

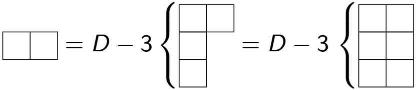
(Irreducible) mixed symmetry tensors (neither completely symmetric nor completely antisymmetric) can be represented by Young diagrams with more than one row and more than one column. We will be interested in two-column diagrams of the form $(N, 1)$:



Convention: first symmetrize over rows, then antisymmetrize over columns. Young diagrams are in one-to-one correspondence with irreps of the permutation group. This gives a complete set of permutation operators over tensor indices.

Mixed-symmetry tensors also appear in the study of electric-magnetic duality of linearized gravity and higher spin fields.

The following irreducible representations of the massless little group $SO(D-2)$ are equivalent:



This can be seen by dualizing the corresponding fields in the physical gauge:

$$h_{ij} = \epsilon_{ik_1..k_{D-3}} t_j{}^{k_1..k_{D-3}} = \epsilon_{ik_1..k_{D-3}} \epsilon_{jl_1..l_{D-3}} t'^{k_1..k_{D-3}l_1..l_{D-3}}$$

and bearing in mind the trace conditions.

Check: the computation of the number of degrees of freedom of these representations coincide (for instance, use the hook length rule in the $SO(D-2)$ representations and then substract the number of independent trace conditions).

When considered as representations of $GL(D,R)$, they are not equivalent. This originates the covariant dual action principles based on tensors of different Young symmetry type (with their corresponding gauge symmetries).

The Curtright action (Curtirght, 1985) is the action principle for a free massless tensor field of mixed symmetry (2,1).
It is a dual formulation of linearized gravity in five dimensions. Check: same number of physical degrees of freedom.
It is constructed by solely relying on the principle of gauge symmetry:

i) one postulates the most general form of the gauge symmetries

$$\delta T_{\alpha_1\alpha_2\beta} = 2\partial_{[\alpha_1}\sigma_{\alpha_2]\beta} + 2\partial_{[\alpha_1}\alpha_{\alpha_2]\beta} - 2\partial_\beta\alpha_{\alpha_1\alpha_2} \quad ; \sigma_{\mu\nu} = \sigma_{\nu\mu}, \alpha_{\mu\nu} = \alpha_{\nu\mu}$$

ii) then one constructs an invariant Lagrangian

$$S[T_{\alpha_1\alpha_1\beta}] = -\frac{1}{6}\int d^5x \left[F_{\alpha_1\alpha_2\alpha_3\beta}F^{\alpha_1\alpha_2\alpha_3\beta} - 3F_{\alpha_1\alpha_2\beta}{}^{\beta}F^{\alpha_1\alpha_2\gamma}{}_{\gamma}\right]$$

The field strength

$$F_{\alpha_1\alpha_2\alpha_3\beta} = 3\partial_{[\alpha_1}T_{\alpha_2\alpha_3]\beta} = \partial_{\alpha_1}T_{\alpha_2\alpha_3\beta} + \partial_{\alpha_2}T_{\alpha_3\alpha_1\beta} + \partial_{\alpha_3}T_{\alpha_1\alpha_2\beta}$$

is only invariant under the $\sigma_{\mu\nu}$ gauge transformations:

$$\delta F_{\alpha_1\alpha_2\alpha_3\beta} = -6\partial_\beta\partial_{[\alpha_1}\alpha_{\alpha_2\alpha_3]}$$

One needs at least two derivatives to construct a fully gauge invariant object:

$$E_{\alpha_1\alpha_2\alpha_3\beta_1\beta_2} = 2F_{\alpha_1\alpha_2\alpha_3[\beta_1,\beta_2]} = 6\partial_{[\beta_2}\partial_{[\alpha_1}T_{\alpha_2\alpha_3]\beta_1]}$$

The action is

$$S = \int d^5x T^{\mu\nu\rho}G_{\mu\nu\rho}$$

and the equation of motion

$$G_{\alpha_1\alpha_2\beta} = 0$$

with

$$G_{\alpha_1\alpha_2\beta} = E_{\alpha_1\alpha_2\beta} + \frac{1}{2}(\eta_{\alpha_1\beta}E_{\alpha_2} - \eta_{\alpha_2\beta}E_{\alpha_1})$$

the analogue of the linearized Einstein tensor in the dual theory.

One may now wonder: is it possible to formulate an action principle for five dimensional linearized gravity where the graviton and its dual field are placed on an equal footing as dynamical variables? What would be the properties of such an action principle?

The two-potential formulation of five dimensional linearized gravity permits to obtain an action principle involving the spatial components of the graviton and its dual field as the dynamical variables.

This requires the resolution of the constraints in the Hamiltonian formalism and the subsequent substitution in the action principle. The idea is to invert one potential as a function of T_{ijk} and then substitute back in the action. The resulting action principle is spatially non-local (Bunster-Henneaux-Hörtner, 2013).

Two-potential formulation of five dimensional linearized gravity

I. Pauli-Fierz

Hamiltonian form of the action principle:

$$S[h_{mn}, \pi^{mn}, n, n_i] = \int d^5x[\pi^{mn}\dot{h}_{mn} - \mathcal{H} - nC - n_i C^i]$$

Hamiltonian:

$$\mathcal{H} = \pi_{mn}\pi^{mn} - \frac{\pi^2}{3} + \frac{1}{4}\partial_r h_{ij}\partial^r h^{ij} - \frac{1}{4}\partial_r h \partial^r h + \frac{1}{2}\partial_m h \partial_n h^{mn} - \frac{1}{2}\partial_m h^{mn}\partial^r h_{rn}$$

Constraints:

$$\begin{aligned} C^i &\equiv -2\partial_j \pi^{ij} = 0 \\ C &\equiv -\Delta h + \partial^m \partial^n h_{mn} = 0 \end{aligned}$$

Gauge symmetries:

$$\begin{aligned} \delta\pi^{ij} &= -\partial^i \partial^j \xi^0 + \delta^{ij}\Delta\xi^0 \\ \delta h_{ij} &= \partial_i \xi_j + \partial_j \xi_i \end{aligned}$$

Resolution of the momentum constraint:

$$\pi^{mn} = \epsilon^{mkqs}\epsilon^{nrtu}\partial_k\partial_r P_{qstu}$$

for a (2,2) Young type potential P_{ijkl} determined up to the ambiguities

$$\delta_1 P_{ijkl} = \chi_{kl[i,j]} + \chi_{ij[k,l]}, \chi = (2,1)$$
$$\delta_2 P_{ijkl} = \frac{1}{4}[\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}]\xi_0$$

δ_1 is an internal invariance
δ_2 induces the gauge transformation on π_{ij}.

Resolution of the Hamiltonian constraint:

$$\partial_m \partial_n j^{mn} = 0; \, h_{mn} = j_{mn} + \partial_m u_n + \partial_n u_m, \quad j^m{}_m = 0$$

$$j_{mn} = \epsilon_{nkst} \partial^k \phi^{st}{}_m + \epsilon_{mkst} \partial^k \phi^{st}{}_n$$

Ambiguities in the definition of the prepotential:

$$\begin{aligned} \delta_1 \phi_{rsm} &= B_{[r} \delta_{s]m} \\ \delta_2 \phi_{mrs} &= \partial_r S_{sm} - \partial_s S_{rm} + \partial_r A_{sm} - \partial_s A_{rm} + 2\partial_m A_{sr} \end{aligned}$$

δ_1 induces the gauge transformation on h_{ij}
δ_2 is an internal invariance.

II. Curtright theory

Hamiltonian form of the action principle:

$$S[T_{ijk}, \pi^{ijk}, m_{ij}, m_k] = \int dt\, d^4x [\pi^{ijk} \dot{T}_{ijk} - \mathcal{H} - m_j \Gamma^j - m_{ij} \Gamma^{ij}]$$

Hamiltonian:

$$\begin{aligned}
\mathcal{H} &= \frac{1}{2}\pi_{ijk}\pi^{ijk} - \frac{1}{2}\pi^{ij}{}_j \pi_{ik}{}^k \\
&\quad + \frac{1}{2}\partial_i T_{jkl}\partial^i T^{jkl} + \partial_i T_{jkl}\partial^j T^{kil} - \partial_i T_{jk}{}^k \partial^i T^{jl}{}_l \\
&\quad - \partial_i T_{jk}{}^k \partial^j T^{li}{}_l - 2\partial_i T_{jl}{}^l \partial^k T^{ij}{}_k - \frac{1}{2}\partial_l T_{ij}{}^l \partial^k T^{ij}{}_k
\end{aligned}$$

Constraints:

$$\begin{aligned}
\Gamma_i &= \Delta T_{ik}{}^k + \partial_j \partial_i T_{jk}{}^k + \partial^j \partial^k T_{jik} = 0 \\
\Gamma^{ij} &= -2\partial_k(\pi^{ijk} + \pi^{kji}) = 0
\end{aligned}$$

The constraints generate the gauge transformations

$$
\begin{aligned}
\delta T_{ijk} &= 2\partial_{[i}\sigma_{j]k} + 2\partial_{[i}\alpha_{j]k} - 2\partial_k\alpha_{ij} \\
\delta\pi^{ijk} &= \partial^i\partial^k(\sigma^{0j} - 3\alpha^{0j}) - \partial^j\partial^k(\sigma^{0i} - 3\alpha^{0i}) \\
&\quad -\delta^{jk}[\partial^i\partial_l(\sigma^{0l} - 3\alpha^{0l}) - \Delta(\sigma^{0i} - 3\alpha^{0i})] + \delta^{ik}[\partial^j\partial_l(\sigma^{0l} - 3\alpha^{0l}) \\
&\quad -\Delta(\sigma^{0j} - 3\alpha^{0j})]
\end{aligned}
$$

Canonical Poisson bracket

$$
\begin{aligned}
\{T_{ijk}(x), \pi^{mnp}(y)\} &= \frac{1}{3}\Big[\delta^m{}_i\delta^n{}_j\delta^p{}_k - \delta^n{}_i\delta^m{}_j\delta^p{}_k \\
&\quad + \frac{1}{2}(\delta^m{}_i\delta^p{}_j\delta^n{}_k - \delta^p{}_i\delta^m{}_j\delta^n{}_k + \delta^p{}_i\delta^n{}_j\delta^m{}_k - \delta^n{}_i\delta^p{}_j\delta^m_k)\Big]\,\delta^{(4)}(x-y)
\end{aligned}
$$

Resolution of the momentum constraint:

$$\partial_i(\pi^{ijk} + \pi^{kji}) = 0$$

Performing similar steps one finds

$$\pi^{ijk} = \epsilon^{ijmn}\epsilon^{krst}\partial_r\partial_m\Psi_{stn}$$

where Ψ_{ijk} is a $(2,1)$ tensor.
The ambiguities are

$$\delta_1\Psi_{rsm} = \partial_r s_{sm} - \partial_s s_{rm} + \partial_r a_{sm} - \partial_s a_{rm} + 2\partial_m a_{sr}$$

with s_{mn} and a_{mn} symmetric and antisymmetric tensor, and

$$\delta_2\Psi_{rsm} = \xi_{[r}\delta_{s]m}$$

δ_1 induces the gauge transformations on π_{ijk}
δ_2 is an internal invariance.

The rôle of these transformations is exchanged with respect to the Pauli-Fierz theory!

Resolution of the Hamiltonian constraint:

Decompose

$$T_{ijk} = t_{ijk} + \theta_{ijk}$$

with ${t_{ij}}^{j} = 0$ so the constraint reads

$$\partial_i \partial_k t^{ijk} = 0$$

It can be solved in terms of a $(2,2)$ prepotential P_{ijkl}

$$t^{ijk} = -\frac{2}{3}\partial_l [2\epsilon^{klab} {P_{ab}}^{ij} + \epsilon^{ilab} {P_{ab}}^{kj} - \epsilon^{jlab} {P_{ab}}^{ki}]$$

with the ambiguities

$$\delta_1 \phi_{rsm} = B_{[r}\delta_{s]m}$$
$$\delta_2 \phi_{mrs} = \partial_r S_{sm} - \partial_s S_{rm} + \partial_r A_{sm} - \partial_s A_{rm} + 2\partial_m A_{sr}$$

δ_1 is an internal invariance, δ_2 induces the gauge transformations on t_{ijk}.

Again, the rôles of the potential transformations are interchanged with respect to the Pauli-Fierz theory.

From both sides one gets the two-potential action:

$$S = \int dt\, d^4x \Big[2\epsilon^{imab}\epsilon^{jncd}\epsilon_{ilxy}\partial_m\partial_n P_{abcd}\partial^l \dot{\phi}^{xy}{}_j$$
$$- \Big(f^{-4}(R_{ij}[P]R^{ij}[P] - \frac{7}{27}R^2[P]) + f^2(2E_{ijk}[\phi]E^{ijk}[\phi] - \frac{3}{2}E_i[\phi]E^i[\phi]) \Big) \Big]$$

R_{ij}, E_{ijk} are the contractions of the “curvature tensors” for the potentials:

$$R_{ijklmn} = 18\partial_{[i}P_{jk][lm,n]}$$

and

$$E_{ijkmn} = 6\partial_{[n}\partial_{[i}T_{jk]m]}$$

III. A deformation of the Curtright action

Is it possible to obtain some qualitative information about the character of deformed action principles (if they exist) from the two-potential formalism? Idea: derive the two-potential form of the five-dimensional Einstein-Hilbert action linearized around de Sitter space-time using a spatially-flat slicing (planar coordinates) and try to introduce the Curtright field and its conjugate momentum by appropriate inversion formulae.

$$ds^2 = -dt^2 + f^2(t)\delta_{ij}dx^i dx^j$$

$f = e^{kt}$, $k = \sqrt{\frac{2\Lambda}{(D-1)(D-2)}}$, $\Lambda > 0$.
This coordinate choice renders the prepotential analysis very similar to the flat case.

The de Sitter metric satisfies

$$R_{\mu\nu\rho\sigma} = \frac{2\Lambda}{(D-1)(D-2)}\left[g_{\mu\rho}g_{\nu\sigma} - g_{\nu\rho}g_{\mu\sigma}\right]$$

which is the maximally symmetric condition $R_{\mu\nu\rho\sigma} \propto (g_{\mu\rho}g_{\nu\sigma} - g_{\nu\rho}g_{\mu\sigma})$.

Prepotentials in five dimensions

We linearize the Einstein-Hilbert action in the ADM formalism

$$S = \int d^D x \, \left[\pi^{ij} \dot{g}_{ij} + N g^{1/2} (R - 2\Lambda) + N g^{-1/2} \left(\frac{1}{D-2} \pi^2 - g_{ik} g_{jl} \pi^{ij} \pi^{kl} \right) + 2 N_i \pi^{ij}{}_{|j} \right]$$

($g = \det(g_{ij})$, $N_i = g_{0i}$, $N = (-g^{00})^{-1/2}$, $\pi = g_{ij}\pi^{ij}$) around a de Sitter background:

$$g_{ij} = \bar{g}_{ij} + h_{ij}, \quad \pi^{ij} = \bar{\pi}^{ij} + p^{ij}$$
$$N = 1 + n, \quad N_i = n_i$$

with

$$\bar{g}_{ij} = f^2(t)\delta_{ij}$$
$$\bar{\pi}^{ij} = \sqrt{\bar{g}}(\bar{g}^{ij}\bar{K} - \bar{K}^{ij}) = -(D-2)kf^{D-3}\delta^{ij}$$

After linearization, the ADM action takes the form

$$S = \int d^D x \left[p^{ij} \dot{h}_{ij} - \mathcal{H} - nC - n_i C^i \right]$$

Hamiltonian:

$$\mathcal{H} = f^{-D+5} p_{ij} p^{ij} - \frac{f^{-D+5}}{D-2} p^2 - 2(D-3) k p_{ij} h^{ij} + khp$$
$$+ f^{D-7} \left[\frac{1}{4} \partial^i h^{jk} \partial_i h_{jk} - \frac{1}{4} \partial_i h \partial^i h + \frac{1}{2} \partial^i h \partial^j h_{ij} - \frac{1}{2} \partial_i h^{ij} \partial^k h_{kj} \right]$$
$$- k^2 f^{D-5} \frac{(D-2)(-2D+6)}{4} h_{ij} h^{ij}$$

Linearized constraints:

$$C = f^{D-5} (\Delta h - \partial_i \partial_j h^{ij}) + 2kpf^2 + f^{D-3} k^2 h (D-2)(D-3) = 0$$
$$C^i = -2 \partial_j p^{ij} + (D-2) f^{D-5} k (2 \partial_k h^{ik} - \partial^i h) = 0$$

Gauge transformations:

$$\begin{aligned} \{\xi C - \xi_m C^m, h_{ij}\} &= \partial_i \xi_j + \partial_j \xi_i - 2kf^2 \delta_{ij} \xi \\ \{\xi C - \xi_m C^m, p_{ij}\} &= f^{D-5} (-\partial^i \partial^j \xi + \delta_{ij} \Delta \xi) + (D-2)(D-3) k^2 f^2 \delta_{ij} \xi \\ &\quad + (D-2) k (\partial_i \xi_j + \partial_j \xi_i - \partial_m \xi^m \delta_{ij}) \end{aligned}$$

In order to solve the constraints it is useful to perform the canonical transformation

$$\begin{aligned} h_{ij} &\mapsto \hat{h}_{ij} = h_{ij} \\ p^{ij} &\mapsto \hat{p}^{ij} = p^{ij} - \frac{D-2}{2} k f^{D-5}(2h^{ij} - \delta^{ij} h). \end{aligned}$$

derived from the generating functional

$$F[h_{ij}, \hat{p}^{ij}] = \int d^{D-1}x \left[\hat{p}^{ij} h_{ij} + \frac{(D-2)}{2} k f^{D-5}(h_{ij} h^{ij} - \frac{1}{2} h^2)\right]$$

as follows:

$$\begin{aligned} \frac{\delta F}{\delta h^{ij}} &\equiv p_{ij} = \hat{p}_{ij} + \frac{D-2}{2} k f^{D-5}(2h_{ij} - \delta_{ij} h) \\ \frac{\delta F}{\delta \hat{p}^{ij}} &\equiv \hat{h}_{ij} = h_{ij} \end{aligned}$$

The action principle reduces then to

$$S[\hat{p}^{ij}, h_{ij}, n, n_i] = \int d^D x \left[\hat{p}^{ij}\dot{h}_{ij} - H - nC - n_i C^i\right], \qquad (0.\text{-}91)$$

Hamiltonian:

$$H = f^{-D+5}\hat{p}_{ij}\hat{p}^{ij} - \frac{f^{-D+5}}{D-2}\hat{p}^2 + 2k\hat{p}_{ij}h^{ij} + f^{D-7}\left[\frac{1}{4}\partial^i h^{jk}\partial_i h_{jk} - \frac{1}{4}\partial_i h \partial^i h \right.$$
$$\left. + \frac{1}{2}\partial^i h \partial^j h_{ij} - \frac{1}{2}\partial_i h^{ij}\partial^k h_{kj}\right]$$

Constraints:

$$C = f^{D-5}(-\partial^i\partial^j h_{ij} + \Delta h) + 2kf^2\hat{p}$$
$$C^i = -2\partial_j \hat{p}^{ij}.$$

The new canonical variable $\hat{p}^{ij}$ transforms as follows:

$$\delta\hat{p}^{ij} = f^{D-5}(-\partial^i\partial^j\xi + \delta^{ij}\Delta\xi)$$

Focus on five dimensions:

The momentum constraint is solved as in the flat case:

$$\hat{p}^{ij} = \epsilon^{iklm}\epsilon^{jnpq}\partial_k\partial_n P_{lmpq}.$$

with the ambiguities

$$\delta P_{abcd} = 2\chi_{cd[b,a]} + 2\chi_{ab[d,c]} + \frac{1}{4}[\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}]\xi,$$
$$\chi_{abc} = -\chi_{bac}, \chi_{[abc]} = 0$$

Substitution of the trace in the scalar constraint yields

$$\Delta h - \partial_i\partial_j h^{ij} + 4f^2 k\Delta P^{ab}{}_{ab} - 8f^2 k\partial_i\partial_j P^{imj}{}_m = 0.$$

The final expression for h_{ij} reads

$$h_{ij} = \partial^k\epsilon_{ikab}\phi^{ab}{}_j + \partial^k\epsilon_{jkab}\phi^{ab}{}_i + \partial_i u_j + \partial_j u_i$$
$$-8kf^2 P_{ikj}{}^k + \frac{4}{3}kf^2\delta_{ij}P_{mn}{}^{mn}. \qquad (0.\text{-}100)$$

The ambiguities in the choice of the prepotential are

$$\delta\phi_{abc} = \partial_a S_{bc} - \partial_b S_{ac} + \partial_a A_{bc} - \partial_b A_{ac} + 2\partial_c A_{ba}$$
$$+B_{[a}\delta_{b]c} - 16kf^2(\tilde{\chi}_{cab} + \tilde{\chi}_{abc})$$
$$\delta u_i = \xi_i + 16kf^2\epsilon_{ibxy}\tilde{\chi}^{bxy} - 2\partial^l\epsilon_{il}{}^{ab}A_{ba}$$

We can now write the action principle in terms of the prepotentials

$$S[\phi_{ijk}, P_{abcd}] = \int dt\ d^4x \Big[2\epsilon^{imab}\epsilon^{jncd}\epsilon_{ilxy}\partial_m\partial_n P_{abcd}\partial^l \dot{\phi}^{xy}{}_j$$
$$+\frac{32}{3}k\dot{P}\partial_a\partial_b P^{ab} - 8k\dot{P}_{ij}\partial_a\partial_b P^{iajb} + 8kf^2\epsilon^{jlab}\partial^i\partial_l\phi_{ab}{}^k\partial_i P_{jk} - 8kf^2\epsilon_{jlab}\partial_i\partial^l\phi^{abi}\partial_k P^{kj}$$
$$+\frac{72}{9}k^2f^2\partial_j P\partial^j P + 32k^2f^2\partial^i P_{ik}\partial_j P^{jk} - 16f^2k^2\partial_i P_{jk}\partial^i P^{jk} - \frac{64}{3}k^2f^2\partial_i P\partial_j P^{ij}$$
$$-\left(f^{-4}(R_{ij}[P]R^{ij}[P] - \frac{7}{27}R^2[P]) + f^2(2E_{ijk}[\phi]E^{ijk}[\phi] - \frac{3}{2}E_i[\phi]E^i[\phi])\right)\Big]$$

$$R_{ijklmn} = 18\partial_{[i}P_{jk][lm,n]}$$

and

$$E_{ijkmn} = 6\partial_{[n}\partial_{[i}T_{jk]m]}$$

We notice that the form of the (linearized) two-potential action principle in a de Sitter background (planar cooridnates) is the same as in the flat case (up to time-dependent factors) with the addition of terms propotional to a power of Λ.

Idea: use a gauge where these extra terms do not appear and then introduce the Curtright variables using the inversion formula already known.

Although the constraints can be solved without prior fixing of the gauge, in order to construct the dual theory it is useful to use the gauge choice $p = 0$:

$$\begin{aligned} \hat{p}^{ij} &= a^{ij} + \delta\hat{p}^{ij} = a^{ij} - \partial^i\partial^j\xi + \delta^{ij}\Delta\xi \\ h_{ij} &= b_{ij} - 2kf^2\xi\delta_{ij} \end{aligned}$$

$a = 0$: achieved through the gauge choice $\xi = \frac{1}{3}\Delta^{-1}\hat{p}$.

The constraints take the same form as in the flat case:

$$\begin{aligned} \partial_j a^{ij} &= 0 \\ \Delta b - \partial^i\partial^j b_{ij} &= 0 \end{aligned}$$

so they are solved

$$\begin{aligned} a^{ij} &= f^{-2}\partial^k\partial^l\epsilon_{ikab}\epsilon_{jlcd}P^{abcd} \\ b_{ij} &= f^2(\partial^l\epsilon_{ilab}\phi^{ab}{}_j + \partial^l\epsilon_{jlab}\phi^{ab}{}_i) + \partial_i u_j + \partial_j u_i \end{aligned} \tag{0.-112}$$

After substituting in the action, the terms proportional to k and k^2 are no longer present:

$$S[P_{ijkl}, \phi_{abc}] = \int dt\ d^4x\ \Big[2\partial_m\partial_k\epsilon^{imnp}\epsilon^{jkst}P_{npst}\partial^l\epsilon_{ilab}\dot{\phi}^{ab}{}_j$$
$$-\Big(f^{-4}(R_{ij}[P]R^{ij}[P] - \frac{7}{27}R^2[P])$$
$$+\ f^2(2E^{ijk}[\phi]E_{ijk}[\phi] - \frac{3}{2}E_i[\phi]E^i[\phi])\Big)\Big]$$

One may now define the canonical pair of dual variables as in the flat case:

$$\begin{aligned}
\hat{t}^{ijk} &= -\frac{2}{3}f^{-2}\partial_l\left[2\epsilon^{klab}P_{ab}{}^{ij} + \epsilon^{ilab}P_{ab}{}^{kj} - \epsilon^{jlab}P_{ab}{}^{ki}\right] \\
\hat{\pi}_{ijk} &= f^2\epsilon_{ijmn}\epsilon_{krst}\partial^m\partial^r\phi^{stn}.
\end{aligned}$$

The action

$$S[\hat{t}_{ijk}, \hat{\pi}_{ijk}, m_j, m_{ij}] = \int d^5x \, \left[\hat{\pi}^{ijk}\dot{\hat{t}}_{ijk} - \mathcal{H} - m_j\Gamma^j - m_{jk}\Gamma^{jk}\right]$$

reproduces the form of the two-potential action, with Hamiltonian density

$$\begin{aligned}
\mathcal{H} = &-2k\hat{\pi}^{ijk}\hat{t}_{ijk} + \frac{1}{2}\partial_i\hat{t}_{jkl}\partial^i\hat{t}^{jkl} + \partial_i\hat{t}_{jkl}\partial^j\hat{t}^{kil} \\
&-\frac{1}{2}\partial^k\hat{t}_{ijk}\partial_l\hat{t}^{ijl} + \frac{1}{2}\hat{\pi}_{ijk}\hat{\pi}^{ijk} - \frac{1}{2}\hat{\pi}_i{}^{ji}\hat{\pi}^k{}_{jk}
\end{aligned}$$

and constraints

$$\begin{aligned}
\Gamma^j &= \partial_i\partial_k\hat{t}^{ijk} \\
\Gamma^{ij} &= -2\partial_k(\hat{\pi}^{ijk} + \hat{\pi}^{kji}).
\end{aligned}$$

This is regarded as the dual theory of the standard action written in terms of the 'new' variables (b_{ij}, a^{ij}).

In order to get the action principle dual to Pauli-Fierz in the untransformed variables (h_{ij}, π^{ij}) one has to undo the canonical transformation in the dual picture: $(\hat{t}_{ijk}, \hat{\pi}^{ijk}) \to (t_{ijk}, \pi^{ijk})$. Express the generating functional in terms of the relevant dual variables.

Since the canonical transformation leaves h_{ij} invariant, it is natural to expect in the dual theory the action of the canonical transformation on $\hat{\pi}_{ijk}$ to be the identity map: we set $\hat{\pi}^{ijk} = \pi^{ijk}$.

We introduce the inversion formulas (valid in the flat case and also in our gauge choice)

$$\phi_{ijk}[\hat{\pi}] = -\frac{1}{2}\Delta^{-1}\hat{\pi}_{ijk}$$

$$P_{abcd}[\hat{t}] = \frac{1}{8}\left[\epsilon_{abij}\partial^i\Delta^{-1}\hat{t}_{cd}{}^j + \epsilon_{cdij}\partial^i\Delta^{-1}\hat{t}_{ab}{}^j\right] - \frac{1}{24}\left[\epsilon_{abij}\partial^i\Delta^{-1}\hat{t}_{cd}{}^j + \epsilon_{cdij}\partial^i\Delta^{-1}\hat{t}_{ab}\right.$$

$$\left. + \epsilon_{caij}\partial^i\Delta^{-1}\hat{t}_{bd}{}^j + \epsilon_{adij}\partial^i\Delta^{-1}\hat{t}_{bc}{}^j + \epsilon_{bcij}\partial^i\Delta^{-1}\hat{t}_{ad}{}^j + \epsilon_{bdij}\partial^i\Delta^{-1}\hat{t}_{ca}{}^j\right]$$

The generating functional reads

$$F[\pi^{ijk}, \hat{t}_{ijk}] \quad = \quad \int d^4x[-\hat{t}^{ijk}\pi_{ijk} - 3k\pi^{ijk}\Delta^{-1}\pi_{ijk}]$$

When expressed in terms of the dual variables, the generating functional depends on the 'old' conjugate momentum π^{ijk} and the 'new' field $\hat{t}_{ijk}$, so the relevant relations are now

$$t_{ijk} = -\frac{\delta F}{\delta \pi^{ijk}} = \hat{t}_{ijk} + 6k\Delta^{-1}\pi_{ijk}, \quad \hat{\pi}^{ijk} = -\frac{\delta F}{\delta \hat{t}_{ijk}} = \pi^{ijk}$$

The action is now expressed in terms of the pair (t_{ijk}, π^{ijk}):

$$S[t_{ijk}, \pi^{ijk}, m_i, m_{ij}] = \int dt d^4x \left[\pi^{ijk}\dot{t}_{ijk} - \mathcal{H} - m_i\Gamma^i - m_{ij}\Gamma^{ij}\right]$$

(getting rid of total time derivatives). The Hamiltonian density

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_\Lambda$$

is the sum of the 'free' Hamiltonian

$$\begin{aligned}\mathcal{H}_0 &= \frac{1}{2}\partial_i t_{jkl}\partial^i t^{jkl} + \partial_i t_{jkl}\partial^j t^{kjl} - \frac{1}{2}\partial^k t_{ijk}\partial_l t^{ijl} \\ &\quad + \frac{1}{2}\pi_{ijk}\pi^{ijk} - \frac{1}{2}\pi_i{}^{ji}\pi^k{}_{jk}\end{aligned}$$

and the term carrying the deformation

$$\mathcal{H}_\Lambda = 4k\pi^{ijk}t_{ijk} - 6k^2\pi^{ijk}\Delta^{-1}\pi_{ijk}$$

The deformed constraints are

$$\begin{aligned}\Gamma^j &= \partial_i\partial_k(t^{ijk} - 6k\Delta^{-1}\pi_{ijk} - \delta_{ijk}) \\ \Gamma^{ij} &= -2\partial_k(\pi^{ijk} + \pi^{kji})\end{aligned}$$

Conclusions

The introduction of a positive cosmological constant in the Pauli-Fierz theory corresponds to the introduction of spatially non-local terms in the dual theory. This is regarded as a deformation of the Curtright action.

Our analysis relies on the choice of planar coordinates, which rendes the analysis similar to the flat case.

Conceptual questions: what is the interpretation of a cosmological constant in the dual theory? Properties of space-time in the dual picture?

Thank you