Ultra-Hyperbolic Equations

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We obtained a classical solution in \textit{Fourier} space to the modified Cauchy problem in $p + q = n = 6$, where $p = q = 3$, with signature $(+, +, +, -, -, -)$. The two extra time-like dimensions along with an additional bounded condition $|u| \leq M$, where $M \in \mathbb{R}$ is an arbitrary constant. We also obtained upper-bound value to the solution.

\[
|\hat{f}(p, \xi)| = M. \tag{1}
\]
Abstract

We obtained a classical solution in Fourier space to the modified Cauchy problem in $p + q = n = 6$, where $p = q = 3$, with signature $(+, +, +, -, -, -)$. The two extra time-like dimensions along with an additional bounded condition $|u| \leq M$, where $M \in \mathbb{R}$ is an arbitrary constant. We also obtained upper-bound value to the solution.

$$
\left| \hat{f}(p, \xi) \right| = M.
$$

We seek the distributional solutions or generalized functions to the following equation,

$$
L^k u = f(x).
$$

The elementary solutions obtained

$$
E = \lim_{\varepsilon \to +0} - \frac{e^{3\pi i}}{4\pi^3} (P + i\varepsilon |x_1^2 + \ldots + x_6^2|)^{-2}
+ \lim_{\varepsilon \to -0} \frac{e^{-3\pi i}}{4\pi^3} (P - i\varepsilon |x_1^2 + \ldots + x_6^2|)^{-2}
$$
Issues with Theories with Extra Time-Like Dimensions

- Closed time-like loops $\Rightarrow$ violations of *causality* [1][Thorne]. Motion of extra time-like dimensions must be non-regular or chaotically quantized $\Rightarrow$ low probability for creating a closed loop.
Issues with Theories with Extra Time-Like Dimensions

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- Violations of *Unitarity* $\Rightarrow$ compactified time dimensions effectively leads to the imaginary correction to the potential $\Rightarrow$ violations of conservation of probability [3][Yndurain]
Issues with Theories with Extra Time-Like Dimensions

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- Violations of *Unitarity* $\Rightarrow$ compactified time dimensions effectively leads to the imaginary correction to the potential $\Rightarrow$ violations of conservation of probability [3][Yndurain]

- Tachyonic modes $\Rightarrow$ instabilities of a given system [2]
Ultra-Hyperbolic Equations (Modified Cauchy Problem) [6][7][5]

\[
\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} = \frac{\partial^2 u}{\partial t_1^2} + \frac{\partial^2 u}{\partial t_2^2} + \frac{\partial^2 u}{\partial t_3^2} \tag{3}
\]

\[u(x, t, 0) = f(x, t), \tag{4}\]

\[u_t(x, t, 0) = g(x, t), \tag{5}\]

with a bounded condition on the solution,

\[|u| \leq M, \tag{6}\]

for \(\forall t \geq 0\), and an arbitrary constant \(M \in \mathbb{R}\).
Ultra-Hyperbolic Equations (Modified Cauchy Problem)

\[ \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} = \frac{\partial^2 u}{\partial t^2_1} + \frac{\partial^2 u}{\partial t^2_2} \]  

(3)

\[ u(x, t, 0) = f(x, t), \]  

(4)

\[ u_t(x, t, 0) = g(x, t), \]  

(5)

with a bounded condition on the solution,

\[ |u| \leq M, \]  

(6)

for \( \forall t \geq 0 \), and an arbitrary constant \( M \in \mathbb{R} \).

Where \( u = u(x, t, t), t = (t_1, t_2), t = \text{physical time}, \) and 
\( x = (x^1, x^2, x^3). \) Using bold variables to represent vector quantities.....
Ultra-Hyperbolic Equations (Modified Cauchy Problem)

- **Fourier** transform over the $x$ and $t$ variables,

\[
\hat{u}(p, \xi, t) = \int_{\mathbb{R}^5} u(x, t, t) \exp i (p \cdot x + \xi \cdot t) \, dx \, dt. \tag{7}
\]

Inverse **Fourier** transform over $p$ and $\xi$,
- Fourier transform over the $x$ and $t$ variables,

\[ \hat{u} (p, \xi , t) = \int_{\mathbb{R}^5} u(x, t, t) \exp i (p \cdot x + \xi \cdot t) \, dx \, dt. \]  

(7)

- Inverse Fourier transform over $p$ and $\xi$,

\[ u(x, t, t) = \frac{1}{(2\pi)^5} \int_{\mathbb{R}^5} \hat{u} (p, \xi , t) \exp -i (p \cdot x + \xi \cdot t) \, dp \, d\xi. \]  

(8)
Ultra-Hyperbolic Equations (Modified Cauchy Problem)

- Fourier transform over the $x$ and $t$ variables,

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Inverse Fourier transform over $p$ and $\xi$,

$$u(x, t, t) = \frac{1}{(2\pi)^5} \int_{\mathbb{R}^5} \hat{u}(p, \xi, t) \exp -i (p \cdot x + \xi \cdot t) \, dp \, d\xi. \quad (8)$$

- Where $p = (p^1, p^2, p^3)$, $\xi = (\xi^1, \xi^2)$, $dx = dx^1 dx^2 dx^3$
  $dt = dt^1 dt^2 dt^3$, $d\xi = d\xi^1 d\xi^2$ and $dp = dp^1 dp^2 dp^3$. 
The Cauchy Problem in Fourier Space is given by:

\[ \hat{u}_{tt}(p, \zeta, t) + (p^2 - \zeta^2)\hat{u}(p, \zeta, t) = 0, \quad (9) \]

\[ \hat{u}(p, \zeta, 0) = \hat{f}(p, \zeta), \quad (10) \]

\[ \hat{u}_t(p, \zeta, 0) = \hat{g}(p, \zeta), \quad (11) \]

where \( \hat{f}(p, \zeta) \) and \( \hat{g}(p, \zeta) \) are functions on \( \mathbb{R}^5 \), and \( (p, \zeta) \in \mathbb{R}^5 \).
Cauchy Problem in Fourier Space

\[ \hat{u}_{tt} (p, \xi, t) + (p^2 - \xi^2) \hat{u} (p, \xi, t) = 0, \]  
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where \( \hat{f} (p, \xi) \) and \( \hat{g} (p, \xi) \) are functions on \( \mathbb{R}^5 \), and \((p, \xi) \in \mathbb{R}^5 \).

Let \( E \subset \mathbb{R}^5, E = \{(p, \xi) \in \mathbb{R}^5 \mid |\xi| > |p|\} \) *Elliptic region*

\( H \subset \mathbb{R}^5, H = \{(p, \xi) \in \mathbb{R}^5 \mid |\xi| < |p|\} \) *Hyperbolic region*

\( N \subset \mathbb{R}^5, N = \{(p, \xi) \in \mathbb{R}^5 \mid |\xi| = |p|\} \) *Linear region*
If \((p, \xi) \in N\), the null region. Then the solution to \(\hat{u}_{tt} (p, \xi, t) = 0\),

\[
\hat{u}_N (p, \xi, t) = \hat{g} (p, \xi) \cdot t + \hat{f} (p, \xi).
\]  

(12)

The bounded condition on \(|\hat{u}_N (p, \xi, t)| = \hat{f} (p, \xi) \leq M\),

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(13)
Solution to the Reduced Cauchy Problem

- If \((p, \xi) \in N\), the \textit{null} region. Then the solution to \(\hat{u}_{tt}(p, \xi, t) = 0\),

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\hat{u}_N(p, \xi, t) = \hat{g}(p, \xi) \cdot t + \hat{f}(p, \xi).
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\[
|\hat{u}_N(p, \xi, t)| = \hat{f}(p, \xi) \leq M.
\] (13)

- If \((p, \xi) \in H\), \textit{hyperbolic} region. Then the solution to \(\hat{u}_{tt}(p, \xi, t) + (p^2 - \xi^2)\hat{u}(p, \xi, t) = 0\).

\[
\hat{u}_H(p, \xi, t) = \hat{f}(p, \xi) \cos(\lambda_H \cdot t) + \hat{g}(p, \xi) \frac{\sin(\lambda_H \cdot t)}{\lambda_H},
\] (14)

where \(\lambda_H = \pm \sqrt{p^2 - \xi^2}\). The solutions are oscillatory...
If \((p, \xi) \in E\), Elliptic region and the bounded condition, \(|u| \leq M\), then the solution to \(\hat{u}_{tt}(p, \xi, t) + (p^2 - \xi^2)\hat{u}(p, \xi, t) = 0\)

\[
\hat{u}_E(p, \xi, t) = \hat{f}(p, \xi) \exp(-\lambda_E \cdot t)
\]  

(15)

where \(\lambda_E = \pm \sqrt{\xi^2 - p^2}\), and \(\lambda_E = i \cdot \lambda_H\).
Solutions to the Reduced Cauchy Problem

If \((p, \xi) \in E\), Elliptic region and the bounded condition, \(|u| \leq M\), then the solution to \(\hat{u}_{tt}(p, \xi, t) + (p^2 - \xi^2)\hat{u}(p, \xi, t) = 0\)

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\hat{u}_E(p, \xi, t) = \hat{f}(p, \xi) \exp(-\lambda_E \cdot t)
\]

where \(\lambda_E = \pm \sqrt{\xi^2 - p^2}\), and \(\lambda_E = i \cdot \lambda_H\).

The bounded condition on the solution \(|u(p, \xi, t)| \leq M \ \forall t \geq 0\) and a bit of analysis, one yields the value of the upper-bound to be

\[
M = |f(x, t)|.
\]
The solution to the modified Cauchy problem in Fourier space, for all \((p, \xi, t) \in \mathbb{R}^6 \ \forall \ t \geq 0\).

\[
\hat{u}(p, \xi, t) = \hat{u}_H(p, \xi, t) + \hat{u}_E(p, \xi, t) + \hat{u}_L(p, \xi, t). \quad (17)
\]
Preliminaries on *Distributions* (Basics of Differential Geometry)[8][9]

- **Tangent bundle** of manifold $M$, $\dim M = n$

\[
TM := \bigcup_{x \in M} T_x M. \tag{18}
\]
Preliminaries on *Distributions* (Basics of Differential Geometry)[8][9]

- **Tangent bundle** of manifold $M$, $\text{dim } M = n$

  \[ TM := \bigcup_{x \in M} T_x M. \]  

- **Vector bundle**: Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $E$ be manifold of dimension $2n$, let $\pi : E \longrightarrow M$ be a surjective map. Let the *fiber* $E_x := \pi^{-1}(x)$ preserve a structure $\mathbb{K}$-**vector space** $V_x$ for each $x \in M$.

  $(E, \pi, M, \{V_x\}_{x \in M})$ is called a $\mathbb{K}$-**vector bundle** if for $\forall x \in M$, $\exists U$ of $x$ and a **diffeomorphism** $\Phi$ or a **local trivialization** of the **vector bundle** $E$.

  \[ \Phi : \pi^{-1}(U) \rightarrow U \times \mathbb{K}^n. \]
If $U = M$, then

$$\Phi : \pi^{-1}(M) \to M \times \mathbb{K}^n.$$  \hfill (20)

Then a vector bundle $E$ is called trivial if it admits a global trivialization (i.e. $U = M$).
If $U = M$, then

$$\Phi: \pi^{-1}(M) \to M \times \mathbb{K}^n.$$  \hfill (20)

Then a vector bundle $E$ is called trivial if it admits a global trivialization (i.e. $U = M$).

Let $\pi_2: U \times \mathbb{K}^n \to \mathbb{K}^n$, then we have a vector space isomorphism that maps element from the fiber $E_x := \pi^{-1}(U)$ to the $\mathbb{K}^n$-vector space

$$\pi_2 \circ \Phi: \pi^{-1}(U) \to \mathbb{K}^n$$  \hfill (21)
Sections and Densities on Vector Bundle

- A section in $E$ is a map $s : M \rightarrow E$ such that $\pi \circ s = id_M$
- The sections in $E = TM$ are called vector fields on $M$ and spanned by $\{\frac{\partial}{\partial x^i}\}_{i=1,..,n}$
- The sections in $E = T^*M$ are called the 1-forms on $M$ and spanned by $\{dx^i\}_{i=1,..,n}$
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The sections in $E = T^*M$ are called the 1-forms on $M$ and spanned by $\{ dx^i \}_{i=1,...,n}$

A section in $E = \Lambda^k T^*M$ is called $k$-forms $dx^{i_1} \wedge \cdots \wedge dx^{i_k}$. When $k = n$ the bundle $\Lambda^n T^*M$ has rank 1 yields

$\implies$ smooth section $dx^1 \wedge \cdots \wedge dx^n \implies M$ being orientable.
Sections and Densities on Vector Bundle

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  $\Rightarrow$ smooth section $dx^1 \wedge \cdots \wedge dx^n \Rightarrow M$ being orientable.

- For each $x \in M$, let $|\Lambda^M|_x$ be a set of all functions $\rho : \Lambda^n T^*_x M \rightarrow \mathbb{R}$ with $\rho (\lambda \xi) = |\lambda| \rho(\xi)$, $\forall \xi \in \Lambda^n T^*_x M$

  $|\Lambda^M|_x$ is 1D vector space $\Rightarrow$ vector bundle $|\Lambda^M|$ of rank 1 over $M$.

  Sections in $|\Lambda^M|$ are called densities are characterized by

  $$|dx| (dx^1 \wedge \cdots \wedge dx^n) = 1 \quad (22)$$
A section in $E$ is a map $s: M \rightarrow E$ such that $\pi \circ s = id_M$

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$\Rightarrow$ smooth section $dx^1 \wedge \cdots \wedge dx^n$ $\Rightarrow$ $M$ being orientable.

For each $x \in M$, let $|\Lambda M|_x$ be a set of all functions $\rho: \Lambda^n T^*_x M \rightarrow R$ with $\rho(\lambda \xi) = |\lambda| \rho(\xi)$ $\forall \xi \in \Lambda^n T^*_x M$

$|\Lambda M|_x$ is 1D vector space $\Rightarrow$ vector bundle $|\Lambda M|$ of rank 1 over $M$.

Sections in $|\Lambda M|$ are called densities are characterized by

$$|dx| (dx^1 \wedge \cdots \wedge dx^n) = 1$$ (22)

$\int_M : \mathcal{D}(M, |\Lambda M|) \rightarrow R$

$\int_M f |dx| = \int_{\varphi(U)} (f \circ \varphi^{-1})(x^1, ..., x^n) dx^1 ... dx^n$
The $\mathbb{K}$-bundle $E \to M$. Let $\mathcal{D}(M, E)$ space of compactly support sections in $E$. 
The $\mathbb{K}$-bundle $E \to M$. Let $\mathcal{D}(M, E)$ space of compactly support sections in $E$.

Equip $E$ with connection $\nabla$, induces a connection $\nabla$ on the dual vector bundle $E^*$ by 

$$(\nabla_X \theta)(s) := \partial_X (\theta(s)) - \theta(\nabla_X s)$$

for all $X \in C^\infty(M, TM)$, $\theta \in C^\infty(M, E^*)$, $s \in C^\infty(M, E)$, and $\theta(s) \in C^\infty(M)$.
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For a continuously differentiable section $\varphi \in C^1(M, E)$, the covariant derivative is a continuous section in $T^*M \otimes E$,

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For a continuously differentiable section $\varphi \in C^1(M, E)$, the covariant derivative is a continuous section in $T^* M \otimes E$,

$$\nabla \varphi \in C^0(M, T^* M \otimes E)$$

In general, if

$\varphi \in C^k(M, E) \to \nabla^k \varphi \in C^0(M, T^* M \otimes \ldots \otimes T^* M \otimes E)$ $k$-times
K-linear map \( T : \mathcal{D}(M, E^*) \rightarrow W \) is called a distribution in \( E \) with values in \( W \). Where \( \mathcal{D}'(M, E, W) \) space of all \( W \)-valued distributions in \( E \).
\textbf{Distributions as Sections of the Vector Bundles} [8]

- $\mathbb{K}$-linear map $T : \mathcal{D}(M, E^*) \to W$ is called a \textit{distribution} in $E$ with values in $W$. Where $\mathcal{D}'(M, E, W)$ space of all $W$-valued distributions in $E$.

- Let $E$ and $F$ be two $\mathbb{K}$-vector bundles over $M$, The linear differential operator $L$
  
  $L : C^\infty(M, E) \to C^\infty(M, F)$, $\varphi \in \mathcal{D}(M, E)$
  
  $L^* : C^\infty(M, F^*) \to C^\infty(M, E)$, $\psi \in \mathcal{D}(M, F^*)$
**Distributions as Sections of the Vector Bundles** [8]

- **K-linear map** $T : \mathcal{D}(M, E^*) \to W$ is called a *distribution* in $E$ with values in $W$. Where $\mathcal{D}'(M, E, W)$ space of all $W$-valued distributions in $E$.

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- Defined by smooth maps $A_\alpha : U \to (M, \text{Hom}_K(E, F))$ such that on $U \ni x$

  $Ls = \sum_{|\alpha| \leq k} A_\alpha \frac{\partial^{|\alpha|} s}{\partial x^\alpha}$ where $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ with

  $|\alpha| = \sum_{r=1}^d \alpha_r \leq k$ and $\frac{\partial^{|\alpha|}}{\partial x^\alpha} = \frac{\partial^{\alpha_1+\ldots+\alpha_d}}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}}$

  $\int_M \psi(L\varphi) dV = \int_M (L^* \psi) \varphi dV$
Distributions as *Sections* of the Vector Bundles [8]

- \( \mathbb{K} \)-linear map \( T : \mathcal{D}(M, E^*) \to W \) is called a *distribution* in \( E \) with values in \( W \). Where \( \mathcal{D}'(M, E, W) \) space of all \( W \)-valued *distributions* in \( E \).

- Let \( E \) and \( F \) be two \( \mathbb{K} \)-vector bundles over \( M \). The linear differential operator \( L \)
  \[ L : C^\infty(M, E) \to C^\infty(M, F), \varphi \in \mathcal{D}(M, E) \]
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- Defined by smooth maps \( A_\alpha : U \to (M, \text{Hom}_\mathbb{K}(E, F)) \) such that on \( U \ni x \)
  \[ Ls = \sum_{|\alpha| \leq k} A_\alpha \frac{\partial^{|\alpha|} s}{\partial x^\alpha} \text{ where } \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d \text{ with } |\alpha| := \sum_{r=1}^d \alpha_r \leq k \text{ and } \frac{\partial^{|\alpha|}}{\partial x^\alpha} = \frac{\partial^{\alpha_1+\ldots+\alpha_d}}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}} \]
  \[ \int_M \psi(L\varphi) dV = \int_M (L^*\psi) \varphi dV \]

- If \( L \) is of order zero, then the operators are just *sections* of the \( \mathbb{K} \)-vector bundle homomorphism \( L \in C^\infty(M, \text{Hom}_\mathbb{K}(E, F)) \).
The principle symbol of $L$ is the map $\sigma_L : T^*M \rightarrow \text{Hom}_K(E, F)$. For every $\zeta = \sum_{r=1}^{d} \zeta_r \cdot dx^r \in T^*_x M$ we get

$$\sigma_L(\zeta) = \sum_{|\alpha|=k} \zeta^\alpha A_\alpha(x), \quad x \in M \quad \text{and} \quad \zeta^\alpha = \zeta^\alpha_1 \ldots \zeta^\alpha_d$$
The principle symbol of $L$ is the map $\sigma_L : T^* M \to \text{Hom}_{\mathbb{K}}(E, F)$. For every $\xi = \sum_{r=1}^{d} \xi_r \cdot dx^r \in T^*_x M$ we get

$$\sigma_L(\xi) = \sum_{|\alpha| = k} \xi^\alpha A_\alpha(x), \ x \in M \text{ and } \xi^\alpha = \xi_{\alpha_1} \cdots \xi_{\alpha_d}$$

- $\text{grad} : C^\infty(M, \mathbb{R}) \to C^\infty(M, TM)$ with principal symbol $\sigma_{\text{grad}}(\xi)f = f \cdot \xi^\#$
- $\text{div} : C^\infty(M, TM) \to C^\infty(M, \mathbb{R})$ with principal symbol $\sigma_{\text{div}}(\xi)X = \xi(X)$
The principle symbol of $L$ is the map $\sigma_L : T^*M \to \text{Hom}_K(E, F)$. For every $\xi = \sum_{r=1}^{d} \xi_r \cdot dx^r \in T^*_x M$ we get $\sigma_L(\xi) = \sum_{|\alpha|=k} \xi^\alpha A_\alpha(x)$, $x \in M$ and $\xi^\alpha = \xi_1^\alpha_1 \ldots \xi_d^\alpha_d$

$\text{grad} : C^\infty(M, \mathbb{R}) \to C^\infty(M, TM)$ with principal symbol $\sigma_{\text{grad}}(\xi)f = f \cdot \xi^\#$

$\text{div} : C^\infty(M, TM) \to C^\infty(M, \mathbb{R})$ with principal symbol $\sigma_{\text{div}}(\xi)X = \xi(X)$

$d : C^\infty(M, \Lambda^k T^*M) \to C^\infty(M, \Lambda^{k+1} T^*)$ with principle symbol $\sigma_d(\xi)\omega = \xi \wedge \omega$

$\nabla : C^\infty(M, E) \to C^\infty(M, T^*M \otimes E) \to \sigma_{\nabla}(\xi)e = \xi \otimes e$
If $E$ is a complex vector bundle, a positive definite quadratic form is given on each fiber $E_x$. Let $\theta : E_\Omega \to \Omega \times \mathbb{C}^r$ be a trivialization and let $(e_1, \ldots, e_r)$ be the corresponding frame field for the fiber $E_\Omega$. The associated inner product is called Hermitian matrix, $h^{\lambda\mu}(x) = \langle e^{\lambda}(x), e^{\mu}(x) \rangle$, (23) $\forall x \in \Omega$. A sesquilinear map $C^\infty_p(M, E) \times C^\infty_q(M, E) \to C^\infty_{p+q}(M, E) \Rightarrow (s, t) \mapsto \{s, t\}$. If $s = \sum \sigma^{\lambda} \otimes e^{\lambda}$ and $t = \sum \tau^{\mu} \otimes e^{\mu}$, and we have $\{s, t\} = \sum_{1 \leq \lambda, \mu \leq r} \sigma^{\lambda} \wedge \tau^{\mu} \langle e^{\lambda}, e^{\mu} \rangle$. The metric tensor on $M$ given by $G = \sum_{i, j} g^{ij} dz_i \otimes dz_j$, $G$ is just a section of $E^* \otimes_R E^*$. I think we are ready to find the generalized solutions...
If $E$ is a *complex vector bundle*, a positive definite quadratic form is given on each fiber $E_x$. Let $\theta : E_\Omega \rightarrow \Omega \times \mathbb{C}^r$ be a *trivialization* and let $(e_1, \ldots, e_r)$ be the corresponding frame field for the fiber $E_\Omega$.

The associated *inner product* is called *Hermitian matrix*,

$$h_{\lambda \mu}(x) = \langle e_\lambda(x), e_\mu(x) \rangle,$$

\(\forall x \in \Omega\). A sesquilinear map $C^\infty_p(M, E) \times C^\infty_q(M, E) \rightarrow C^\infty_{p+q}(M, E)$

\(\Rightarrow (s, t) \mapsto \{s, t\}\).

If $s = \sum \sigma_\lambda \otimes e_\lambda$ and $t = \sum \tau_\mu \otimes e_\mu$, and we have

$$\{s, t\} = \sum_{1 \leq \lambda, \mu \leq r} \sigma_\lambda \wedge \overline{\tau}_\mu \langle e_\lambda, e_\mu \rangle.$$
If $E$ is a complex vector bundle, a positive definite quadratic form is given on each fiber $E_x$. Let $\theta : E_\Omega \to \Omega \times \mathbb{C}^r$ be a trivialization and let $(e_1, \ldots, e_r)$ be the corresponding frame field for the fiber $E_\Omega$.

The associated inner product is called Hermitian matrix,

$$h_{\lambda \mu}(x) = \langle e_\lambda(x), e_\mu(x) \rangle,$$ (23)

$\forall x \in \Omega$. A sesquilinear map $C^\infty_p(M, E) \times C^\infty_q(M, E) \to C^\infty_{p+q}(M, E)$

$\Rightarrow (s, t) \mapsto \{s, t\}$.

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The metric tensor on $M^h$ given by $G = \sum_{i,j} g_{ij} dz_i \otimes d\overline{z}_j$, $G$ is just a section of $E^* \otimes \mathbb{R} E^*$. 

---

"Complex Vector Bundle and Complex manifolds [11][12]"
If $E$ is a complex vector bundle, a positive definite quadratic form is given on each fiber $E_x$. Let $\theta : E_\Omega \to \Omega \times \mathbb{C}^r$ be a trivialization and let $(e_1, \ldots, e_r)$ be the corresponding frame field for the fiber $E_\Omega$.

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I think we are ready to find the generalized solutions...
Since the *quadratic form* is invariant under linear transformations

\[ P = x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2 \]  

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we seek the elementary solution \( E(x) = f(P) \)

\[ L^k E = \delta(x) \]  \hspace{1cm} (25)
Invariant *Quadratic Form* under *Linear Transformations* [10]

- Since the *quadratic form* is invariant under linear transformations
  \[
P = x_1^2 + ... + x_p^2 - x_{p+1}^2 - ... - x_{p+q}^2
\]

- we seek the elementary solution \( E(x) = f(P) \)
  \[L^k E = \delta(x)\]

- \( f(P) = (P + i0)^{\lambda} + (P - i0)^{\lambda} \) where
  \((P \pm i0)^{\lambda} = \lim_{\varepsilon \to 0^\pm} (P \pm i\varepsilon |x_1^2 + ... + x_n^2|)^{\lambda}\)
  are homogeneous generalized functions to \( P \). The solutions are in \( \mathbb{C} \)-vector bundle \( TM \)
  with \( \lambda = -\frac{1}{2} n + k \), \( f(P) \) is the solution to \( L^k E = \delta(x) \).
Let $\mathbb{P} = \sum_{i,j} g_{ij} x_i x_j$ be a quadratic form whose coefficients may be complex

$$\mathbb{P} = P_1 + iP_2$$

where $P_2$ is positive definite.
Let $P = \sum_{i,j} g_{ij} x_i x_j$ be a quadratic form whose coefficients may be complex

$$P = P_1 + iP_2$$  \hspace{1cm} (26)

where $P_2$ is positive definite.

The $P^\lambda$ can be written as $P^\lambda = \exp \lambda (\ln |P| + i \arg P)$, where $0 < \arg P < \pi$, for single valued analytic function of $\lambda$. Start with $P = \sum_{i,j} g_{ij} x_i x_j = iP_2$ lies on positive imaginary axis

$\Rightarrow g_{ij} = ia_{ij}$ for $i, j = 1, \ldots n$. We find singular points of $P^\lambda$ on the upper-half-plane,
Generalized Functions with Complex Coefficients

\[
\left( \mathcal{P}^\lambda, \phi \right) = e^{\frac{1}{2} \pi \lambda i} \int \left( \sum_{i,j} g_{ij} x_i x_j \right)^\lambda \phi \, dx. \quad (27)
\]

We can always find a linear transformations \( x_i = \sum_{j=1}^n \alpha_{ij} x'_j \) such that \( \sum_{i,j} g_{ij} x_i x_j \) transforms into \( r^2 = x'_1^2 + \cdots + x'_n^2 \), and \( \phi \in E^* \) dual to \( K \)-vector bundle and is a test function.
Generalized Functions with Complex Coefficients

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(\mathbb{P}^\lambda, \phi) = e^{\frac{1}{2} \pi i \lambda} \int \left( \sum_{i,j} g_{ij} x_i x_j \right)^\lambda \phi dx.
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We can always find a linear transformations \( x_i = \sum_{j=1}^n \alpha_{ij} x'_j \) such that \( \sum_{i,j} g_{ij} x_i x_j \) transforms into \( r^2 = x'_1^2 + \cdots + x'_n^2 \), and \( \phi \in E^* \) dual to \( \mathbb{K}\text{-vector bundle} \) and is a test function.

\[
(\mathbb{P}^\lambda, \phi) = \frac{e^{\frac{1}{2} \pi i \lambda} i}{\sqrt{|a|}} \int 2^\lambda \phi dx' = \frac{e^{\frac{1}{2} \pi i \lambda} i}{\sqrt{(-i)^n |g|}} \int r^{2\lambda} \phi dx' \]

(28)

where \( |g| \) is determinant of coefficients of \( \mathbb{P} \).
Generalized Functions with Complex Coefficients

\[
\text{res}_{\lambda = -\frac{1}{2}n} \mathbb{P}^\lambda = \frac{e^{-\frac{1}{4} \pi q i} \pi^{\frac{n}{2}}}{\sqrt{(-i)^n |g| \Gamma\left(\left(\frac{n}{2}\right)\right)}} \delta(x) \tag{29}
\]
Generalized Functions with Complex Coefficients

\[ \text{res} \quad \mathbb{P}^\lambda_{\lambda=-\frac{1}{2} n} = \frac{e^{-\frac{1}{4} \pi q_i \pi \frac{n}{2}}}{\sqrt{(-i)^n |g| \Gamma((\frac{n}{2}))}} \delta(x) \] (29)

To calculate the residues of \( \mathbb{P}^\lambda \) at others singular points, we consider

\[ L_{\mathbb{P}} = \sum g^{ij} \frac{\partial^2}{\partial x_i \partial x_j} \] (30)

defined by \( \sum_{i=1}^{n} g^{ri} g_{is} = \delta^r_s \) where \( \delta^r_s = 1 \) for \( r = s \) and \( \delta^r_s = 0 \) otherwise.
We have \( L_P P^{\lambda+1} = 4(\lambda + 1)(\lambda + \frac{n}{2})P^\lambda \)
\( \Rightarrow L_P^k P^{\lambda+k} = 4^k (\lambda + 1) \ldots (\lambda + k) \ldots (\lambda + \frac{n}{2} + k - 1)P^\lambda. \)
Generalized Functions with Complex coefficients

- We have $L_P \mathbb{P}^{\lambda+1} = 4(\lambda + 1)(\lambda + \frac{n}{2})\mathbb{P}^\lambda$
  - $\Rightarrow L_P^k \mathbb{P}^{\lambda+k} = 4^k(\lambda + 1)\ldots(\lambda + k)\ldots(\lambda + \frac{n}{2} + k - 1)\mathbb{P}^\lambda$.

- Yields $\mathbb{P}^\lambda = \frac{1}{4^k(\lambda+1)\ldots(\lambda+k)\ldots(\lambda + \frac{n}{2} + k - 1)} L_P^k \mathbb{P}^{\lambda+k}$.
We have $L_P P^{\lambda + 1} = 4(\lambda + 1)(\lambda + \frac{n}{2})P^\lambda$
\[\Rightarrow L^k_P P^{\lambda + k} = 4^k(\lambda + 1)(\lambda + k)...(\lambda + \frac{n}{2} + k - 1)P^\lambda.\]

Yields $P^\lambda = \frac{1}{4^k(\lambda + 1)(\lambda + k)...(\lambda + \frac{n}{2} + k - 1)} L^k_P P^{\lambda + k}.$

With this
\[
\text{res}_{\lambda = -\frac{1}{2}n - k} P^\lambda = \frac{1}{4^k(\lambda + 1)(\lambda + k)...(\lambda + \frac{n}{2} + k - 1)} \big|_{\lambda = -\frac{1}{2}n - k} \text{res}_{\lambda = -\frac{1}{2}n} L^k_P P^\lambda
\]
We get
\[
\text{res}_{\lambda = -\frac{1}{2}n - k} P^\lambda = \frac{e^{-\frac{1}{4}\pi q} \pi^{\frac{n}{2}}}{4^k k! \sqrt{(-i)^n |g| \Gamma(\frac{n}{2} + k)}} L^k_P \delta(x). \quad (31)
\]
To find residue at other singular points

\[ \text{res}_{\lambda=-\frac{1}{2}n-k} P^\lambda = \frac{e^{-\frac{1}{4} \pi qi \pi \frac{n}{2}}}{\sqrt{(-i)^n |g| \Gamma(\frac{n}{2})}} L^k \delta(x) \].

This is equation assume \( P \) lies on the positive imaginary axis. We continue it analytically to the entire upper half-plane.
Generalized Functions with Complex coefficients

- To find residue at other singular points
  \[ \text{res} \quad \mathbb{P}^\lambda = \frac{e^{-\frac{1}{4} \pi q_i \pi \frac{n}{2}}}{\sqrt{(-i)^n |g| \Gamma((\frac{n}{2})^n)}} L^k \delta(x). \]  
  This is equation assume \( \mathbb{P} \) lies on the positive imaginary axis. We continue it analytically to the entire upper half-plane.

- \( \mathbb{P} = P_1 + iP_2 \) where \( P_1 \) and \( P_2 \) are quadratic forms with real coefficients and \( P_2 \) is positive definite. \( \exists \) a nonsingular transformation \( x_i = \sum_{j=1}^{n} b_{ij} y_j \) such that \( P_1 = \lambda_1 y_1^2 + ... + \lambda_n y_n^2 \), \( P_2 = y_1^2 + ... + y_n^2 \) thus we have
  \[ |g| = |b|^2(\lambda_1 + i)...(\lambda_n + i) \]
  \[ \Rightarrow (-i)^n |g| = |b|^2(1 - i\lambda_1)...(1 - i\lambda_n) \]
  \[ \sqrt{(-i)^n |g|} = |b| (1 - i\lambda_1)^{\frac{1}{2}}... (1 - i\lambda_n)^{\frac{1}{2}} \]
\( P^\lambda \) is regular analytic function of \( \lambda \) the upper-half-plane except for \( \lambda = -\frac{1}{2} n - k \), \( k \) is non-negative integer. We need to calculate the residues at the singular points by
\( \mathbb{P}^\lambda \) is regular analytic function of \( \lambda \) the upper-half-plane except for \( \lambda = -\frac{1}{2}n - k \), \( k \) is non-negative integer. We need to calculate the residues at the singular points by

\[
\text{res}_{\lambda=-\frac{1}{2}n-k} \mathbb{P}^\lambda = \frac{e^{-\frac{1}{4}\pi q i \pi \frac{n}{2}}}{\sqrt{(-i)^n |g| \Gamma\left(\frac{n}{2}\right)}} \left( \sum_{i,j=1} g^{ij} \frac{\partial^2}{\partial x_i \partial x_j} \right)^k \delta(x)
\]  (32)
The residue $\text{res}_{\lambda = -\frac{1}{2}n-k} (P_1 + i0)^\lambda = \lim_{P_2 \to 0 \lambda = -\frac{1}{2}n-k} \text{res}_{\lambda = -\frac{1}{2}n-k} (P_1 + iP_2)^\lambda$ where

$$P_2 = \epsilon |x_1^2 + \ldots + x_n^2| \quad \epsilon > 0, \quad P_1 = P$$
The residue \( \text{res}_{\lambda=-\frac{1}{2}n-k} (P_1 + i0)^\lambda = \lim_{P_2 \to 0, \lambda=-\frac{1}{2}n-k} \text{res}_{\lambda} (P_1 + iP_2)^\lambda \) where

\[
P_2 = \epsilon \left| x_1^2 + \ldots + x_n^2 \right| \quad \epsilon > 0, \quad P_1 = P
\]

\[
\lim_{\epsilon \to 0} \sqrt{(-i)^n |g|} = |b| (\epsilon - i\lambda_1)^{\frac{1}{2}} \ldots (\epsilon - i\lambda_n)^{\frac{1}{2}}
\]

now recall that there are \( p \) of eigenvalues of \( P_1 \) are positive and \( q \) eigenvalues are negative.
The residue \( \text{res}_{\lambda=-\frac{1}{2}n-k} (P_1 + i0)^\lambda = \lim_{P_2 \to 0} \text{res}_{\lambda=-\frac{1}{2}n-k} (P_1 + iP_2)^\lambda \) where

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\[ \lim_{\epsilon \to 0} \sqrt{(-i)^n |g|} = \sqrt{|\lambda_1 \ldots \lambda_n|} (-i)^{\frac{p}{2}} (i)^{\frac{q}{2}} . \]
The residue $\text{res} \ (P_1 + i0)^\lambda = \lim_{\lambda \to 0} \text{res} \ (P_1 + iP_2)^\lambda$ where

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$$\lim_{\epsilon \to 0} \sqrt{(-i)^n |g|} = \sqrt{|\lambda_1 \ldots \lambda_n|} (-i)^{\frac{p}{2}} (i)^{\frac{q}{2}}.$$

For $\lambda = -\frac{1}{2}n - k$, we have

$$\text{res} \ (P \pm i0)^\lambda = \frac{e^{\mp \frac{1}{2} \pi q i} \pi^n}{4^k k! \sqrt{|\lambda| \Gamma\left(\frac{n}{2} + k\right)}} \left( \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} \right)^k \delta(x). \quad (33)$$
We seek the *distributional* solutions or *generalized functions*

\[ L^k u = f(x) \]  \hspace{1cm} (34)

where \( L = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \ldots - \frac{\partial^2}{\partial x_{p+q}^2} \), when \( k = 1 \), we have *ultra-hyperbolic* PDE equation. When either \( p = 1 \) or \( q = 1 \), we have the *wave equation*, and either \( p = 0 \) or \( q = 0 \), we have the *Laplace’s equation*.
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Since the *quadratic form* is invariant under any linear transformations

\[ P = x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2 \]  \((35)\)
We seek the *distributional* solutions or *generalized functions*

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P = x_1^2 + ... + x_p^2 - x_{p+1}^2 - ... - x_{p+q}^2
\]  \hspace{1cm} (35)

- We seek the *elementary solution* \( E(x) = f(P) \) to the \( L^k E = \delta(x) \).
We seek the distributional solutions or generalized functions

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where \( L = \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - ... - \frac{\partial^2}{\partial x_{p+q}^2}, \) when \( k = 1, \) we have ultra-hyperbolic PDE equation. When either \( p = 1 \) or \( q = 1, \) we have the wave equation, and either \( p = 0 \) or \( q = 0, \) we have the Laplace’s equation.

Since the quadratic form is invariant under any linear transformations

\[ P = x_1^2 + ... + x_p^2 - x_{p+1}^2 - ... - x_{p+q}^2 \]  \hspace{1cm} （35）

We seek the elementary solution \( E(x) = f(P) \) to the \( L^k E = \delta(x). \)

\[ f(P) = (P + i0)^\lambda + (P - i0)^\lambda \] where \( (P \pm i0)^\lambda = \lim_{\epsilon \to +0} (P \pm i\epsilon \left| x_1^2 + ... + x_n^2 \right|)^\lambda \) are homogeneous generalized functions to \( P. \) The solutions are in \( \mathbb{C}-\)vector bundle \( TM \) with \( \lambda = -\frac{1}{2} n + k, \) \( f(P) \) is the solution to \( L^k E = \delta(x). \)
\[ L^k(P + i0)^{\lambda+k} = 4^k(\lambda + 1)...(\lambda + k)(\lambda + \frac{n}{2})...(\lambda + \frac{n}{2} + k - 1) = (P + i0)^\lambda. \]
\( L^k(P + i0)^{\lambda + k} = 4^k(\lambda + 1)\ldots(\lambda + k)(\lambda + \frac{n}{2})\ldots(\lambda + \frac{n}{2} + k - 1) = P + i0)^{\lambda}. \)

\( L^k(P + i0)^{-\frac{n}{2} + k} = \\ 4^k(1 - \frac{n}{2})\ldots(k - \frac{n}{2})(-\frac{n}{2} + \frac{n}{2})\ldots(k - 1)! \frac{e^{-\frac{1}{2}\pi qi\pi \frac{n}{2}}}{\Gamma((\frac{n}{2}))} \delta(x). \)
\[ L^k(P + i0)^{\lambda + k} = 4^k(\lambda + 1) \cdots (\lambda + k)(\lambda + \frac{n}{2}) \cdots (\lambda + \frac{n}{2} + k - 1) = (P + i0)^\lambda. \]

\[ L^k(P + i0)^{-\frac{n}{2} + k} = 4^k(1 - \frac{n}{2}) \cdots (k - \frac{n}{2})(-\frac{n}{2} + \frac{n}{2}) \cdots (k - 1)! \frac{e^{-\frac{1}{2} \pi q i \pi^\frac{n}{2}}}{\Gamma((\frac{n}{2}))} \delta(x). \]

\[ L^k(P + i0)^{-\frac{n}{2} + k} e^{\frac{1}{2} \pi q i \Gamma(\frac{n}{2} - k)} \frac{n}{4^k(k-1)!\pi^\frac{n}{2}} (-1)^k = \delta(x). \]
Hence

\[ E_1 = (-1)^k \frac{e^{\frac{1}{2} \piqi}}{4^k (k - 1)!} \pi \frac{n}{2} \Gamma\left( \frac{n}{2} - k \right) (P + i0)^{-\frac{n}{2} + k} \]  

(36)
Hence

\[ E_1 = (-1)^k e^{\frac{1}{2} \pi i q_i} \Gamma\left(\frac{n}{2} - k\right) \frac{\pi^{n/2}}{4^k (k - 1)!} (P + i0)^{-\frac{n}{2}+k} \]  

(36)

Similarly,

\[ E_2 = (-1)^k e^{-\frac{1}{2} \pi i q_i} \Gamma\left(\frac{n}{2} - k\right) \frac{\pi^{n/2}}{4^k (k - 1)!} (P - i0)^{-\frac{n}{2}+k} \]  

(37)
Hence

\[ E_1 = (-1)^k \frac{e^{\frac{1}{2} \pi q i} \Gamma\left(\frac{n}{2} - k\right)}{4^k (k - 1)! \pi^{\frac{n}{2}}} (P + i0)^{-\frac{n}{2} + k} \]  

(36)

Similarly,

\[ E_2 = (-1)^k \frac{e^{-\frac{1}{2} \pi q i} \Gamma\left(\frac{n}{2} - k\right)}{4^k (k - 1)! \pi^{\frac{n}{2}}} (P - i0)^{-\frac{n}{2} + k}. \]  

(37)

Thus the solution \( E = E_1 - E_2 \)

\[ \text{res}_{\lambda = -\frac{1}{2} n - k} (P \pm i0)^\lambda = \frac{e^{\pm \frac{1}{2} \pi q i} \pi^{\frac{n}{2}}}{4^k k! \sqrt{|\lambda|} \Gamma\left(\frac{n}{2} + k\right)} \left( \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} \right)^k \delta(x) \]  

(38)
The elementary solution to $L^k u = f(x)$
The elementary solution to $L^k u = f(x)$

$$= (-1)^k \frac{e^{\frac{1}{2} \pi q i} \Gamma\left(\frac{n}{2} - k\right)}{4^k (k-1)! \pi^{\frac{n}{2}}} (P + i0)^{-\frac{n}{2} + k} - (-1)^k \frac{e^{-\frac{1}{2} \pi q i} \Gamma\left(\frac{n}{2} - k\right)}{4^k (k-1)! \pi^{\frac{n}{2}}} (P - i0)^{-\frac{n}{2} + k}$$
The elementary solution to $L^k u = f(x)$

$$= (-1)^k \frac{e^{\frac{1}{2} \pi qi \Gamma(n_2 - k)}}{4^k(k-1)! \pi^{\frac{n}{2}}} (P + i0)^{-\frac{n}{2} + k} - (-1)^k \frac{e^{-\frac{1}{2} \pi qi \Gamma(n_2 - k)}}{4^k(k-1)! \pi^{\frac{n}{2}}} (P - i0)^{-\frac{n}{2} + k}$$

where $L = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_4^2} - \frac{\partial^2}{\partial x_5^2} - \frac{\partial^2}{\partial x_6^2}$, for $n = 6, p = q = 3$, and $k = 1$.

The solutions reduces to

$$E = -\frac{e^{\frac{3}{2} \pi i}}{4\pi^3} (P + i0)^{-2} + \frac{e^{-\frac{3}{2} \pi i}}{4\pi^3} (P - i0)^{-2}. \quad (39)$$

Therefore, our homogeneous generalized functions or distributions are sections of the $\mathbb{K}^6$-vector bundle.
References:


