Applied Newton-Cartan Geometry

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why non-relativistic gravity?
The Holographic Principle

Gravity is not only used to describe the gravitational force!
Condensed Matter Physics

Examples: liquid helium, cold atomic gases and quantum Hall fluids

Effective Field Theory (EFT) coupled to NC gravity ⇒ universal features

c ompare to

Coriolis force

Supersymmetric Localization

supersymmetry allows to apply powerful localization techniques to exactly calculate partition functions of (non-relativistic) supersymmetric field theories

Pestun (2007); Festuccia, Seiberg (2011),

This should also apply to the non-relativistic case!
Non-relativistic Gravity

- Free-falling frames: *Galilean symmetries*

- Earth-based frame: *Newtonian gravity/Newton potential*

- *no* frame-independent formulation
Geometry

for a frame-independent formulation you need geometry

Riemann (1867)
Einstein achieved two things in 1915:

- He made gravity consistent with special relativity
- He used a frame-independent formulation
General Relativity

- Free-falling frames: Poincare symmetries

- arbitrary frames: metric tensor field
Newton-Cartan Gravity

- Newton-Cartan (NC) gravity is Newtonian gravity in arbitrary frame

- NC gravity contains more gravitational fields than Newton potential
Outline

Gauging
Outline

Gauging

Null-reduction

Further Developments
Outline

Gauging

Null-reduction

Limits

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Non-relativistic Gravity

In the non-relativistic case free-falling frames are connected by the Galilean symmetries:

- time translations: \( \delta t = \xi^0 \)
- space translations: \( \delta x^i = \xi^i \quad i = 1, 2, 3 \)
- spatial rotations: \( \delta x^i = \lambda^i_{\ j} x^j \)
- Galilean boosts: \( \delta x^i = \lambda^i t \)
(kinematics of) Newton-Cartan gravity follows from ‘gauging’ the Bargmann algebra
Relativistic versus Non-relativistic Massive Particle

\[ S_{\text{relativistic}}(\text{massive}) = -m \int d\tau \sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} \quad \mu = 0, 1, 2, 3 \]

Lagrangian is invariant under Poincare symmetries

\[ S_{\text{non-relativistic}}(\text{massive}) = \frac{m}{2} \int \frac{\dot{x}^i \dot{x}^j \delta_{ij}}{t} d\tau \quad i = 1, 2, 3 \]

Lagrangian is not invariant under Galilean boosts:

\[ \delta L_{\text{non-relativistic}}(\text{massive}) = \frac{d}{d\tau} (mx^i \chi^j \delta_{ij}) \quad \Rightarrow \]

modified Noether charge gives rise to central extension!
‘Gauging’ the Bargmann algebra

Andringa, Panda, de Roo + E.B. (2011)

\[ [J_{ab}, P_c] = -2\delta_c [a P_b] , \quad [J_{ab}, G_c] = -2\delta_c [a G_b] , \]
\[ [G_a, H] = -P_a , \quad [G_a, P_b] = -\delta_{ab} Z , \quad a = 1, 2, \ldots, d \]

<table>
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<th>symmetry</th>
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<th>gauge field</th>
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<tr>
<td>time translations</td>
<td>( H )</td>
<td>( \tau_\mu )</td>
<td>( \mathcal{R}_{\mu\nu}(H) )</td>
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<tr>
<td>space translations</td>
<td>( P^a )</td>
<td>( e^{a}_\mu )</td>
<td>( \mathcal{R}^{a}_{\mu\nu}(P) )</td>
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<tr>
<td>Galilean boosts</td>
<td>( G^a )</td>
<td>( \omega^{a}_\mu )</td>
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<td>spatial rotations</td>
<td>( J^{ab} )</td>
<td>( \omega^{ab}_\mu )</td>
<td>( \mathcal{R}^{ab}_{\mu\nu}(J) )</td>
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<tr>
<td>central charge transf.</td>
<td>( Z )</td>
<td>( m_\mu )</td>
<td>( \mathcal{R}_{\mu\nu}(Z) )</td>
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Imposing Constraints

\[
\mathcal{R}_{\mu\nu}^a(P) = 0, \quad \mathcal{R}_{\mu\nu}(Z) = 0 : \text{ solve for spin-connection fields}
\]

\[
\mathcal{R}_{\mu\nu}(H) = \partial_{[\mu} \tau_{\nu]} = 0 \rightarrow \tau_{\mu} = \partial_{\mu} \tau : \text{ absolute time (‘zero torsion’)}
\]

\[
\mathcal{R}_{\mu\nu}^{ab}(J) \neq 0 : \text{ un-constrained off-shell}
\]

\[
\mathcal{R}_{\mu\nu}^a(G) \neq 0 : \text{ un-constrained off-shell}
\]
The Transformation Rules

The independent NC fields \( \{ \tau_\mu, e_\mu^a, m_\mu \} \) transform as follows:

\[
\begin{align*}
\delta \tau_\mu &= 0, \\
\delta e_\mu^a &= \lambda^a_b e_\mu^b + \lambda^a \tau_\mu, \\
\delta m_\mu &= \partial_\mu \sigma + \lambda_a e_\mu^a
\end{align*}
\]

The spin-connection fields \( \omega_\mu^{ab} \) and \( \omega_\mu^a \) are functions of \( e, \tau \) and \( m \)
The NC Equations of Motion

The NC equations of motion are given by

\[ \tau^\mu e^\nu_a R_{\mu\nu}^a(G) = 0 \]
\[ e^\nu_a R_{\mu\nu}^{ab}(J) = 0 \quad a + (ab) \]

- after gauge-fixing and assuming flat space the first NC e.o.m. becomes \( \triangle \Phi = 0 \)
- there is no known action that gives rise to these equations of motion
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Kaluza-Klein Reduction

\[ x^M = \{x^\mu, y\}, \quad M = (\mu, y), \quad A = (a, y) \]

non-null Killing vector: \( \xi = \partial_y \)

Reduction Ansatz

\[ E_M^A = \begin{pmatrix} a e^{-1/(D-2)\phi} e_\mu^a & y e^\phi A_\mu \\ \dot{y} & 0 & e^\phi \end{pmatrix} \]

Truncation: \( \phi = 1, \quad A_\mu = 0 \quad \Rightarrow \quad \text{Einstein gravity} \)
Null-reduction

Julia, Nicolai (1994)

\[ x^M = \{x^\mu, v\}, \quad M = (\mu, v), \quad A = (a, +, -) \]

null Killing vector \( \xi = \partial_v : \quad \xi^2 = \xi^M \xi^N G_{MN} = 0 \quad \Rightarrow \quad G_{vv} = 0 \)

Reduction Ansatz (only e.o.m.)

\[ E_M^A = \begin{pmatrix} a & - & + \\ \mu & e_\mu^a & \tau_\mu & -m_\mu \\ v & 0 & 0 & 1 \end{pmatrix} \]

\[ \Lambda_{AB} \quad \Rightarrow \quad \Lambda_{ab} \text{ (spatial rotations) and } \Lambda_{-a} \text{ (Galilean boosts)} \]
Arbitrary Torsion

Off-shell we obtain the transformation rules of NC gravity with arbitrary torsion, i.e. $\tau_{\mu\nu} \equiv \partial_{[\mu} \tau_{\nu]} \neq 0$, leading to modified spin-connections:

$$\hat{\omega}_{\mu}^{\ ab} = \omega_{\mu}^{\ ab} - m_{\mu} \tau^{ab}, \quad \hat{\omega}_{\mu}^{\ a+} = \omega_{\mu}^{\ a} + m_{\mu} \tau_{0}^{a}$$

On-shell Julia and Nicolai obtain additional conditions leading to zero torsion.

We prefer to work off-shell and next truncate:

Torsion $\Rightarrow$ twistless torsion: $\tau_{[\mu} \partial_{\nu} \tau_{\rho]} = 0 \rightarrow$ zero torsion: $\partial_{[\mu} \tau_{\nu]} = 0$
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Inönü Wigner Contraction I

\[ [P_A, M_{BC}] = 2 \eta_{A[B} P_{C]} , \quad [M_{AB}, M_{CD}] = 4 \eta_{[A[C} M_{D]B]} \]

\[ P_0 = \frac{1}{2\omega} H, \quad A = (0, a) \]

\[ P_a = P_a , \quad M_{ab} = J_{ab} , \quad M_{a0} = \omega \, G_a \]

Taking the limit \( \omega \to \infty \) gives the Galilei algebra:

\[ [P_a, G_b] = 0 \]
Inönü Wigner Contraction II

\[
\left[ P_A, M_{BC} \right] = 2 \eta_{A[B} P_{C]}, \quad \left[ M_{AB}, M_{CD} \right] = 4 \eta_{[A[C} M_{D]B]} \quad \text{plus} \quad Z
\]

\[
P_0 = \frac{1}{2\omega} H + \omega Z, \quad Z = \frac{1}{2\omega} H - \omega Z, \quad A = (0, a)
\]

\[
P_a = P_a, \quad M_{ab} = J_{ab}, \quad M_{a0} = \omega G_a
\]

Taking the limit \( \omega \to \infty \) gives the Bargmann algebra including \( Z \):

\[
\left[ P_a, G_b \right] = \delta_{ab} Z
\]
‘Gaugings’, Contractions and Non-relativistic Limits

\[ \text{Poincare } \otimes \text{U}(1) \quad \xrightarrow{\text{‘gauging’}} \quad \text{General relativity } \otimes \text{U}(1) \]

\[ \downarrow \quad \text{contraction} \quad \downarrow \quad \text{non-relativistic limit} \]

\[ \text{Bargmann} \quad \xrightarrow{\text{‘gauging’}} \quad \text{Newton-Cartan gravity} \]
The Massive Non-relativistic Limit

Dautcourt (1964)

add a vector field $M_\mu$ to general relativity with $\partial_{[\mu} M_{\nu]} = 0$

**STEP I:** express relativistic fields $\{E_\mu^A, M_\mu\}$ in terms of non-relativistic fields $\{\tau_\mu, e_\mu^a, m_\mu\}$

$$E_\mu^0 = \omega \tau_\mu + \frac{1}{2\omega} m_\mu, \quad M_\mu = \omega \tau_\mu - \frac{1}{2\omega} m_\mu, \quad E_\mu^a = e_\mu^a \quad \Rightarrow$$

$$E_\mu^a = e_\mu^a - \frac{1}{2\omega^2} \tau^\mu e^\rho_a m_\rho + O(\omega^{-4}) \text{ and similar for } E_\mu^0$$
The Massive Non-relativistic Limit II

STEP II: take the limit $\omega \rightarrow \infty$ in e.o.m. \[ \Rightarrow \]

- the NC transformation rules are obtained
- the NC equations of motion are obtained (but no action!)

Note: the standard textbook limit gives Newton gravity
Other Limits

- massive non-relativistic limit: Klein-Gordon $\rightarrow$ Schrödinger

- massless non-relativistic ‘Galilei’ limit
  Gomis, Rollier, Rosseel, ter Veldhuis + E.B., in preparation; Gomis, Townsend (2016)

- massless ultra-relativistic ‘Carroll’ limit
  
  - flat space holography
    for a review with many references, see Banks and Fischler (2016)
  
  - warped CFT
    Hofman and Rollier (2014)
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Extensions

• **Schrödinger gravity** → NC gravity and **Hořava-Lifshitz gravity**
  
  Hartong, Obers (2015)

  Afshar, Mehra, Parekh, Rollier + E.B. (2015)

• **3D Extended Bargmann Gravity**

  Rosseel + E.B. (2016)

• **3D Higher Spin Extensions**

  Golkar, Nguyen, Roberts, Son (2016)

  Gruemiller, Prohazka, Rosseel + E.B. (2016)

  Medina, Revoy (1985)
the supersymmetric extension of General Relativity was discovered precisely 40 years ago but today there is no known supersymmetric extension of the 4D Newton’s Poisson equation!

Freedman, Ferrara, van Nieuwenhuizen (1976)

only known result: 3D Newton-Cartan supergravity

Andringa, Rosseel, Sezgin + E.B. (2013)

Easiest way to construct 4D NC Supergravity:

- Null-Reduction
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Applications!
March 6-10, 2017:  Save the Date!

Simons Workshop on Applied Newton-Cartan Geometry

organized by Gary Gibbons, Rob Leigh, Djordje Minic, Dam Thanh Son + E.B.