Matrix exponentials, SU(N) group elements, and real polynomial roots

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Abstract

The exponential of an $N \times N$ matrix can always be expressed as a matrix polynomial of order $N - 1$. In particular, a general group element for the fundamental representation of SU($N$) can be expressed as a matrix polynomial of order $N - 1$ in a traceless $N \times N$ hermitian generating matrix, with polynomial coefficients consisting of elementary trigonometric functions dependent on $N - 2$ invariants in addition to the group parameter. These invariants are just angles determined by the direction of a real $N$-vector whose components are the eigenvalues of the hermitian matrix. Equivalently, the eigenvalues are given by projecting the vertices of an $(N - 1)$-simplex onto a particular axis passing through the center of the simplex. The orientation of the simplex relative to this axis determines the angular invariants and hence the real eigenvalues of the matrix.

“Let us revisit Euclid. Let us discover for ourselves a few of the newer results. Perhaps we may be able to recapture some of the wonder and awe that our first contact with geometry aroused.” — H S M Coxeter

Curtright and Zachos (CZ) wrote a brief summary [1] of essential elementary features for the triplet representation [2] of SU(3), thereby distilling some older results [3, 4, 5]. Here, I show how their main results may be extended to almost any $N \times N$ matrix, thereby embellishing [3, 6, 7]. A matrix polynomial form for an exponentiated matrix can always be expressed succinctly in terms of a “response function” that encodes the eigenvalues of the matrix.

In particular, I show how the CZ results may be extended to the fundamental representation of SU($N$) for any $N$. I show that a polynomial form for any group element in the fundamental representation can be expressed in terms of a response function that encodes the real eigenvalues of the hermitian matrix that generates the group element. In addition, I provide a clear geometrical picture of the relevant group invariants in terms of the elementary properties of an $(N - 1)$-simplex [8], as a simple generalization of Viète’s venerable results for the real roots of a cubic equation [9]. I give specific results for SU($N \leq 5$).

The exponential of an $N \times N$ matrix $M$ can be written as a matrix polynomial as a consequence of the Cayley-Hamilton theorem [10], or more directly, as the result of the Lagrange-Sylvester projection matrix method [11]. A reasonably compact polynomial form is given by

$$\exp (itM) = \sum_{n=0}^{N-1} M^n E_n (t) ,$$

$$E_n (t) = \sum_{m=0}^{N-1-n} (-1)^m S_m [\lambda] \left( -i \frac{d}{dt} \right)^{N-1-n-m} F (t) .$$

The invariant functions appearing in the $E_n$ coefficients are given by symmetric polynomials,

$$S_m [\lambda] \equiv \sum_{1 \leq k_1 < k_2 < \cdots < k_m \leq N} \lambda_{k_1} \lambda_{k_2} \cdots \lambda_{k_m} ,$$

where $\lambda_k$ for $k = 1, \cdots, N$ are the eigenvalues of $M$, and by various derivatives of the response function for the matrix, $F (t)$. The latter is defined in terms of the characteristic function $C (z)$ as

$$F (t) = \sum_{k=1}^{N} \frac{\exp (i \lambda_k t)}{\lambda_k} , \quad C (z) = \det (zI - M) ,$$

where I have assumed the eigenvalues are non-degenerate. For degenerate eigenvalues, appropriate limits must be taken.

Alternatively, the symmetric polynomials can be written as invariant traces, hence computed directly from $M$ without knowing the individual eigenvalues [12].

$$S_m [\lambda] = \frac{1}{m!} T_m ,$$
\[ \mathcal{I}_m = \det \begin{pmatrix} \text{tr} (M) & m - 1 & 0 & \cdots & 0 & 0 & 0 \\ \text{tr} (M^2) & \text{tr} (M) & m - 2 & \cdots & 0 & 0 & 0 \\ \text{tr} (M^3) & \text{tr} (M^2) & \text{tr} (M) & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \text{tr} (M^{m-2}) & \text{tr} (M^{m-3}) & \text{tr} (M^{m-4}) & \cdots & \text{tr} (M) & 2 & 0 \\ \text{tr} (M^{m-1}) & \text{tr} (M^{m-2}) & \text{tr} (M^{m-3}) & \cdots & \text{tr} (M^2) & \text{tr} (M) & 1 \\ \text{tr} (M^m) & \text{tr} (M^{m-1}) & \text{tr} (M^{m-2}) & \cdots & \text{tr} (M^3) & \text{tr} (M^2) & \text{tr} (M) \end{pmatrix} . \] (6)

This form of the invariants follows from the generating function for the polynomials as given by a determinant,

\[ \det (I + tM) = \sum_{m=0}^{N} t^m S_m . \] (7)

The result \( \mathfrak{I} \) may also be derived in a straightforward way just by taking the inverse Laplace transform of the resolvent for a general complex \( N \times N \) matrix \( M \), as expressed by

\[ \frac{1}{I - sM} = \sum_{n=0}^{N-1} M^n R_n (s) , \quad R_n (s) = \frac{s^n \text{Trunc} \left[ \det (1 - sM) \right]}{\text{det} (1 - sM)} . \] (8)

where \( \text{Trunc} \left[ f (s) \right] \equiv \sum_{m=0}^{n} \frac{1}{m!} f^{(m)} (0) s^m \) is the \( k \)-th order Taylor polynomial for the function \( f \). This result for the resolvent is well-known \[13, 14, 15\]. In this approach the response function is encountered as

\[ F (t) = \int e^{itz} C (z) \; dz , \] (9)

where the counter-clockwise contour integral \( \oint \) encloses all the eigenvalues of \( M \).

To be more specific I now give some results appropriate to the group \( SU (N) \), for various \( N \), that evince most of the general features. In these specific cases \( I \) is always the \( N \times N \) unit matrix and \( \mathcal{H} \) is a traceless, hermitian matrix, so the eigenvalues are all real. In addition, \( F_N (t) \) is the response function for the \( N \times N \) matrix \( \mathcal{H} \), as defined above. The corresponding characteristic function and its derivative are

\[ C (z) = \prod_{k=1}^{N} (z - \lambda_k) , \quad C' (z) = \sum_{k=1}^{N} \left( \prod_{m=1}^{N} (z - \lambda_m) \right) , \quad C' (\lambda_k) = \prod_{m=1 \atop m \neq k}^{N} (\lambda_k - \lambda_m) . \] (10)

Again, I will assume the eigenvalues are non-degenerate. Otherwise, for the expressions to follow appropriate limits must be taken.

For the \( SU (2) \) doublet, as is well-known,

\[ \exp (it\mathcal{H}) = \left[ \mathcal{H} - iI \frac{d}{dt} \right] F_2 (t) . \] (11)

For the \( SU (3) \) triplet, as given in \( \mathfrak{I} \),

\[ \exp (it\mathcal{H}) = \left[ \mathcal{H}^2 - iH^2 \frac{d}{dt} - I \left( \frac{1}{2} \text{tr} (\mathcal{H}^2) + \frac{d^2}{dt^2} \right) \right] F_3 (t) . \] (12)

For the \( SU (4) \) quartet,

\[ \exp (it\mathcal{H}) = \left[ \mathcal{H}^3 - iH^3 \frac{d}{dt} - \mathcal{H} \left( \frac{1}{2} \text{tr} (\mathcal{H}^2) + \frac{d^2}{dt^2} \right) + i \left( -\frac{1}{3} \text{tr} (\mathcal{H}^3) + \frac{1}{2} i \text{tr} (\mathcal{H}^2) \frac{d}{dt} + i \frac{d^3}{dt^3} \right) \right] F_4 (t) . \] (13)

And finally, for the \( SU (5) \) quintet,

\[ \exp (it\mathcal{H}) = \left[ \mathcal{H}^4 - iH^4 \frac{d}{dt} - \mathcal{H} \left( \frac{1}{2} \text{tr} (\mathcal{H}^2) + \frac{d^2}{dt^2} \right) + \left( \frac{1}{3} \text{tr} (\mathcal{H}^3) \frac{d}{dt} + \frac{1}{2} i \text{tr} (\mathcal{H}^2) \frac{d^2}{dt^2} + \frac{d^3}{dt^3} \right) + i \left( \frac{1}{5} (\text{tr} (\mathcal{H}^2))^2 - \frac{1}{4} \text{tr} (\mathcal{H}^4) + \frac{1}{3} \text{tr} (\mathcal{H}^3) \frac{d}{dt} + \frac{1}{2} \text{tr} (\mathcal{H}^2) \frac{d^2}{dt^2} + \frac{d^4}{dt^4} \right) \right] F_5 (t) . \] (14)
Etc. Note how the form of the result for $SU(N)$ can be immediately obtained from that for $SU(N+1)$ by discarding the unit matrix term for the latter, by decrementing the exponents of the remaining matrix powers of $H$ (but do not change the exponents inside invariant traces), and by replacing $F_{N+1} \rightarrow F_N$.

Conversely, the result for $SU(N)$ can be obtained from that for $SU(N-1)$ just by replacing $F_{N-1} \rightarrow F_N$, by incrementing the powers of $H$ in the $SU(N-1)$ expression, and finally by adding the appropriate unit matrix term. That unit matrix term for $SU(N)$ can always be expressed as a series of derivatives of $F_N$ with “Viète coefficients” given by invariant traces, as described above, namely,

$$I \times (-1)^{N-1} \sum_{n=0}^{N-1} \frac{1}{n!} \mathcal{I}_n \left( \frac{d}{dt} \right)^{N-1-n} F_N(t) .$$

The sum here can also be written as an inverse Laplace transform:

$$\mathcal{L}^{-1} \left[ 1 - (-is)^N \frac{\det(H)}{\det(I - isH)} \right] = (-1)^{N-1} \sum_{n=0}^{N-1} \frac{1}{n!} \mathcal{I}_n \left( \frac{d}{dt} \right)^{N-1-n} F_N(t) .$$

The explicit form of the $\mathcal{I}_k$ invariants for all $k$ are given in [10] as traced powers of $H$. For the specific examples it suffices to note: $\mathcal{I}_0 = 1$, $\mathcal{I}_1 = \text{tr}(H)$, $\mathcal{I}_2 = (\text{tr}(H))^2 - \text{tr}(H^2)$, $\mathcal{I}_3 = (\text{tr}(H))^3 - 3\text{tr}(H)\text{tr}(H^2) + 2\text{tr}(H^3)$, and $\mathcal{I}_4 = (\text{tr}(H))^4 - 6(\text{tr}(H))^2\text{tr}(H^2) + 8\text{tr}(H)\text{tr}(H^3) + 3(\text{tr}(H^2))^2 - 6\text{tr}(H^4)$, where $\text{tr}(H) = 0$ for $SU(N)$ generators.

Thus a compact form of the fundamental exponential polynomial for any $SU(N)$ can be obtained by using [15] to construct sequentially the hierarchy of polynomials for $SU(M \leq N)$, starting from the trivial result $I$ for $SU(1)$ [18]. However, more work is required to obtain explicit results for the response functions since the eigenvalues for a generic $N \times N$ matrix, whether hermitian or not, pose some challenges [19], especially in the limit of large $N$.

Nevertheless, as an explicit example, an element of an $SU(2)$ subgroup of $SU(N)$ is always manageable in closed form, where $N = 2j+1$ fixes the spin of the embedded $SU(2)$ representation. The standard choice for the generator $\vec{J}$ of a rotation about axis $\vec{n}$ gives a characteristic function

$$C(\lambda) = \det \left( \lambda I - \vec{n} \cdot \vec{J} \right) = \prod_{k=0}^{2j} \left( \lambda - (j-k) \right) = \frac{\Gamma \left( \lambda + j + 1 \right)}{\Gamma \left( \lambda - j \right)} .$$

The trace norm of this standard generator is given by a product of the quadratic $su(2)$ Casimir, $j(j+1)$, and the matrix rank, $N = 2j+1$, namely,

$$\text{tr} \left[ \left( \vec{n} \cdot \vec{J} \right)^2 \right] = \sum_{k=0}^{2j} (j-k)^2 = \frac{1}{3} j(j+1)(2j+1) .$$

Traces of all odd powers of $\vec{n} \cdot \vec{J}$ vanish, of course, since the eigenvalues are symmetrically distributed about zero. Traces of all even powers are easily computed, as given by the following generating function (i.e. the character of the group element, upon replacing $x \rightarrow i\theta$ where $\theta$ is the rotation angle):

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \text{tr} \left[ (\vec{n} \cdot \vec{J})^n \right] = \sum_{k=0}^{2j} e^{x(j-k)} = \frac{\sinh \left( (2j+1) x/2 \right)}{\sinh (x/2)} = \left( 2j+1 \right) \left( 1 + \frac{x^2}{6} j(j+1) + \frac{x^4}{360} j(j+1)(3j(j+1)-1) + O(x^6) \right) .$$

Note that $\frac{1}{2j+1} \text{tr} \left[ (\vec{n} \cdot \vec{J})^{2k} \right]$ is always a $k$th order polynomial in the quadratic Casimir.

More importantly for the exponentiated matrix, the response function for the embedded $SU(2)$ spin $j$ representation is

$$F_{N=2j+1}(\theta) = \sum_{k=0}^{2j} \frac{\exp(i(j-k)\theta)}{C'(j-k)} = \frac{\sin \left( \frac{1}{2j} \sin \frac{2(j+1)}{2} \theta / \theta \right)}{(2j)!} .$$

Derivatives of this response function are readily evaluated to obtain a matrix polynomial expansion for $\exp \left( i\theta \vec{n} \cdot \vec{J} \right)$, although it takes quite a bit of additional work to reduce the resulting polynomial to the compact form in [16] [17].
For generic SU (N) generators the eigenvalues and the response function cannot be so easily determined as was the case for the embedded SU (2) example. However, there is an elegant geometrical way to parameterize the eigenvalues that generalizes the well-known picture developed by Viète in the 16th century for cubic equations with three real roots [9]. Indeed, the discussion to follow is easily adapted to provide a visualization of the solution to any Nth order equation with only real roots, although it does not immediately give an explicit solution for the roots in terms of traces and/or determinants.

For SU (N) the N real eigenvalues of the traceless hermitian generator H may be envisioned as the components of a Euclidean vector, \( \vec{\lambda} \in \mathbb{E}_N \). This vector of eigenvalues lies in an \((N - 1)\)-dimensional hyperplane that passes through the origin of the Euclidean space, with normal \( \vec{n} \equiv (1, 1, \cdots, 1) \in \mathbb{E}_N \), as defined by \( 0 = \vec{n} \cdot \vec{\lambda} = \sum_{k=1}^{N} \lambda_k = \text{tr} (H) \).

The vector \( \vec{\lambda} \) is also a point on a sphere \( S_{N-1} \subset \mathbb{E}_N \), with the non-vanishing radius of the sphere defined by the matrix invariant \( r^2 \equiv \vec{\lambda} \cdot \vec{\lambda} = \sum_{k=1}^{N} \lambda_k^2 = \text{tr} (H^2) \). The intersection of this hyperplane and \((N - 1)\)-sphere define a particular \((N - 2)\)-sphere in the eigenvalue space, \( S_{N-2} \subset \mathbb{E}_N \). The eigenvalues of \( H \) therefore comprise the components of a vector \( \vec{\lambda} \) that is a point on this \( S_{N-2} \), and for a given radius \( r \) the vector \( \vec{\lambda} \) may be completely specified by the standard \( N - 2 \) spherical polar angles parameterizing \( S_{N-2} \).

For example, for SU (3) the vector of eigenvalues is a point on the circle defined by the intersection of the aforementioned plane and 2-sphere. The components of \( \vec{\lambda} \) are specified by the radius of the circle and by a single angle \( \theta \), as noted long ago by Viète for cubic equations with three real roots. Thus

\[
\lambda_k = \sqrt{\frac{2}{3}} r \cos \left( \theta + 2\pi k/3 \right) \quad \text{for} \quad k = 1, 2, 3,
\]

where \( \prod_{k=1}^{3} \lambda_k = 1/\sqrt{6} \) and that the eigenvalues may also be viewed as the projections of three points equally spaced on the circle. That is to say, the eigenvalues are given by projecting along a particular axis the vertices of an equilateral triangle (i.e. a 2-simplex) circumscribed by a circle, where the axis of interest is a diameter of the circle. The angle \( \theta \) is determined by the orientation of the triangle with respect to this axis.

Perhaps even more surprisingly, this geometrical picture generalizes completely to any \( N \). The eigenvalues of any generator for the fundamental representation of SU \( (N) \) are given by projecting the \( N \) vertices of an \((N - 1)\)-simplex onto a particular axis that passes through the center of the simplex. The orientation of the simplex with respect to the axis determines the \( N - 2 \) angles that are needed, in addition to the radius \( r \), to specify the components of \( \vec{\lambda} \). I now explain in detail how these geometrical facts are established, if they are not already obvious.

The components of \( \vec{\lambda} \) are the eigenvalues of \( H \). Each of these components is given by a Euclidean inner product with one of the unit vectors that define the standard orthonormal basis for \( \mathbb{E}_N \). Thus

\[
\lambda_k = \vec{e}_k \cdot \vec{\lambda} \, , \quad \text{where} \quad \vec{e}_1 = (1, 0, \cdots, 0) \, , \quad \vec{e}_2 = (0, 1, \cdots, 0) \, , \cdots \, , \quad \vec{e}_N = (0, \cdots, 0, 1) \, .
\]

On the other hand, these \( N \) unit vectors are the vertices of the \( N \)-cell that defines the standard \((N - 1)\)-simplex, embedded in \( \mathbb{E}_N \). This standard simplex lies in a hyperplane “parallel” to the one that contains the vector of eigenvalues, with a normal again given by the aforementioned \( \vec{n} \). Moreover, since \( \vec{n} \cdot \vec{\lambda} = 0 \), the inner products with \( \vec{\lambda} \) are unchanged if each of the \( \vec{e}_k \) is translated by adding any amount of the normal \( \vec{n} \). Thus we may rigidly translate the standard simplex so that it too is centered on the origin and lies in the hyperplane containing the eigenvalues. Upon doing so, we obtain a simplex with vertices \( \vec{e}_k - 1/N \vec{n} \), where the eigenvalues are now given by inner products \( \lambda_k = (\vec{e}_k - 1/N \vec{n}) \cdot \vec{\lambda} \). More explicitly,

\[
\vec{e}_1 - 1/N \vec{n} = \left( \frac{N-1}{N}, \frac{1}{N}, \cdots, \frac{1}{N} \right) \, , \quad \vec{e}_2 - 1/N \vec{n} = \left( -\frac{1}{N}, \frac{N-1}{N}, \cdots, -\frac{1}{N} \right) \, , \cdots \, , \quad \vec{e}_N - 1/N \vec{n} = \left( -\frac{1}{N}, \cdots, -\frac{1}{N}, \frac{N-1}{N} \right) \, .
\]

In addition, we may rescale these vertices to obtain a simplex that also lies on the \((N - 2)\)-sphere that contains the vector of eigenvalues. Thus take

\[
\vec{f}_k = \left( \vec{e}_k - 1/N \vec{n} \right) r \sqrt{\frac{N}{N-1}} \, .
\]

One readily checks that these \( N \) vectors are indeed the vertices of a simplex, with

\[
\vec{f}_k \cdot \vec{f}_k = r^2 \quad \text{for any} \quad k = 1, 2, \cdots, N \quad \text{and} \quad \vec{f}_k \cdot \vec{f}_m = \frac{1}{1 - N} r^2 \quad \text{for any} \quad k \neq m \, .
\]
This simplex corresponds to the array of weight vectors \([21]\) for the fundamental representation of \(SU(N)\). The eigenvalues are now obtained by projecting the vertices of this final \((N-1)\)-simplex onto an axis defined by the direction of \(\hat{\lambda}\), i.e. by projecting the vertices onto a particular diameter of the \((N-2)\)-sphere.

Thus, with unit vector \(\vec{e} = \hat{\lambda}/r\),

\[
\lambda_k = \sqrt{\frac{N-1}{N}} \, f_k \cdot \vec{e}. \tag{26}
\]

But note, what matters for the eigenvalues is the relative orientation of the simplex with respect to the axis. Thus we may produce the same eigenvalues if we choose the axis to be in any convenient direction, and appropriately orient an equivalent simplex with respect to the chosen axis.

For example, for the eigenvalues of the \(SU(3)\) generator, as given in \([21]\), we may choose the axis as the horizontal on the plane, with \(\theta\) the planar angle measured counterclockwise from the horizontal. We may then visualize the eigenvalues as projections onto the horizontal axis of the vertices of an equilateral triangle in the plane — a 2-simplex — those vertices being located at angles \(\theta\) and \(\theta \pm 2\pi/3\), and at a distance \(\sqrt{\frac{2}{3}} \, r\) from the origin.

For other examples, consider \(SU(4)\) and \(SU(5)\), and note that the freedom to re-orient jointly the axis \(\text{and}\) the simplex, as if they comprised a single rigid body, leads to apparently different — but nevertheless equivalent — parameterizations of the eigenvalues in terms of the standard spherical polar angles for \(S_{N-2}\).

For \(SU(4)\) the vector of eigenvalues is a point on a 2-sphere. The four eigenvalues, i.e. the components of the four-vector \(\lambda\), may be specified in terms of two polar angles.

\[
\begin{align*}
\lambda_1 &= r \left( -\frac{1}{\sqrt{2}} \sin \phi \sin \theta - \frac{1}{2} \cos \theta \right), \tag{27} \\
\lambda_2 &= r \left( +\frac{1}{\sqrt{2}} \sin \phi \sin \theta - \frac{1}{2} \cos \theta \right), \tag{28} \\
\lambda_3 &= r \left( -\frac{1}{\sqrt{2}} \cos \phi \sin \theta + \frac{1}{2} \cos \theta \right), \tag{29} \\
\lambda_4 &= r \left( +\frac{1}{\sqrt{2}} \cos \phi \sin \theta + \frac{1}{2} \cos \theta \right). \tag{30}
\end{align*}
\]

These are just projections of the vertices of a regular tetrahedron onto an axis through its center, whose orientation is left as an exercise for the reader \([22]\), but which is easily visualized in three Euclidean dimensions.

For \(SU(5)\) the vector of eigenvalues is a point on a 3-sphere. The five eigenvalues, i.e. the components of the five-vector \(\lambda\), may be specified in terms of three polar angles.
\[
\begin{align*}
\lambda_1 &= r \left( -\frac{4}{2\sqrt{5}} \cos \psi \right), \\
\lambda_2 &= r \left( \frac{1}{2\sqrt{5}} \cos \psi - \frac{3}{2\sqrt{3}} \cos \theta \sin \psi \right), \\
\lambda_3 &= r \left( \frac{1}{2\sqrt{5}} \cos \psi + \frac{1}{2\sqrt{3}} \cos \theta \sin \psi - \frac{2}{\sqrt{6}} \cos \phi \sin \theta \sin \psi \right), \\
\lambda_4 &= r \left( \frac{1}{2\sqrt{5}} \cos \psi + \frac{1}{2\sqrt{3}} \cos \theta \sin \psi + \frac{1}{\sqrt{6}} \cos \phi \sin \psi \sin \psi - \frac{1}{\sqrt{2}} \sin \theta \sin \phi \sin \psi \right), \\
\lambda_5 &= r \left( \frac{1}{2\sqrt{5}} \cos \psi + \frac{1}{2\sqrt{3}} \cos \theta \sin \psi + \frac{1}{\sqrt{6}} \cos \phi \sin \theta \sin \psi + \frac{1}{\sqrt{2}} \sin \theta \sin \phi \sin \psi \right).
\end{align*}
\]

These are just projections of the vertices of a pentatope onto an axis through its center. In this case I have chosen things so that the relative orientations of the axis and the five vertices are easily determined in terms of the angular parameterization, even if the entire geometry is not so easily visualized as in the SU(4) example.

Finally, just as the eigenvalues are readily expressed in terms of the radius and the various angles, so too are the latter expressed in terms of combinations of the eigenvalues. But it is not easy — there is no royal road to find the roots of quintic and higher order equations — to express the eigenvalues, or the corresponding angles, in terms of the invariant traces or the determinant of \( H \) for SU(\( N > 4 \)).

For example, for SU(4) as parameterized above,

\[
\begin{align*}
\text{tr} (H) &= \sum_{k=1}^{4} \lambda_k = 0, \quad \text{tr} (H^2) = \sum_{k=1}^{4} \lambda_k^2 = r^2, \\
\text{tr} (H^3) &= \sum_{k=1}^{4} \lambda_k^3 = \frac{3}{4} r^3 \sin \theta \sin (2\theta) \cos (2\phi), \\
\det (H) &= \prod_{k=1}^{4} \lambda_k = \frac{1}{8} \left( (\text{tr} (H^2))^2 - 2 \, \text{tr} (H^4) \right) \\
&= \frac{1}{16} r^4 \left( 1 + (2 \sin^2 \phi - 3) \sin^2 \theta \right) \left( 1 + (2 \cos^2 \phi - 3) \sin^2 \theta \right). 
\end{align*}
\]

This leads to equations cubic and quadratic in \( \sin^2 \theta \) and \( \sin^2 \phi \) that can be solved in closed form in terms of elementary functions of the traces of \( H^2, H^3, \) and \( H^4 \) or \( \det (H) \). For detailed discussions on how to solve the resulting equations, see [6, 23].

For SU(5) as parameterized above, it is straightforward to check that

\[
\begin{align*}
\text{tr} (H^3) &= r^3 \left( \frac{3}{\sqrt{3}} (\cos \psi) \left( \frac{1}{2} - \cos^2 \psi \right) + \frac{5}{2\sqrt{3}} (\sin^3 \psi \cos \theta) \left( \frac{3}{5} - \cos^2 \theta \right) + \frac{2\sqrt{2}}{\sqrt{3}} (\sin^3 \psi \sin^3 \theta \cos \phi) \left( \frac{3}{4} - \cos^2 \phi \right) \right),
\end{align*}
\]

etc. However, in this case I can not find an elementary expression of the angles, or eigenvalues, in terms of invariant traces of powers of \( H \). But then, neither can anyone else, as is well-known [24].

Nonetheless, at the very least I hope the constructions described here for any \( N \) may serve as elementary examples that illustrate the elegant interplay between group theory and geometry. As H S M Coxeter once said [5],

"... in four or more dimensions, we can never fully comprehend them by direct observation. In attempting to do so, however, we seem to peep through a chink in the wall of our physical limitations, into a new world of dazzling beauty. Such an escape from the turbulence of ordinary life will perhaps help to keep us sane."

Acknowledgement  I thank Professor Curtright for focusing my attention on [1], and for suggesting that the simple geometrical construction so succinctly described therein would generalize to any SU(\( N \)).

References

[1] T L Curtright and C K Zachos, “Elementary results for the fundamental representation of SU(3)” eprint: [arxiv:1508.00868 math.RT], Earlier versions were first available from Research Gate.


[12] This is a well-known fact. For example, see Section 3.1 in T L Curtright and D B Fairlie, “A Galileon Primer” eprint: arXiv:1212.6972 [hep-th].


[18] The same procedure may be used to construct sequentially a hierarchy of polynomials for the exponentials of almost any matrix, as in [11].


[20] Although the eigenvalues themselves still define a circle with the same radius as before, namely $r^2$, since $\sum_{k=1}^{3} \cos^2 (\theta + 2\pi k/3) = 3/2$.


[22] Here is the solution to the exercise: The tetrahedron’s vertices are at a distance $\sqrt{3} r/2$ from the origin, such that $\lambda_1 = \frac{1}{2} r (-1, 1, -1) \cdot \hat{e}$, $\lambda_2 = \frac{1}{2} r (-1, -1, 1) \cdot \hat{e}$, $\lambda_3 = \frac{1}{2} r (1, -1, -1) \cdot \hat{e}$, and $\lambda_4 = \frac{1}{2} r (1, 1, 1) \cdot \hat{e}$, where $\hat{e} \equiv (\cos \theta, \sin \theta \cos (\phi + \pi/4), \sin \theta \sin (\phi + \pi/4))$.
