Minimal area surfaces and Wilson loops in the AdS/CFT correspondence

M. Kruczenski

Purdue University

+ recent work w/ A. Irrgang, arXiv:1507.02787
and w/ Changyu Huang, Yifei He (to appear)

Miami 2015
Summary

- **Introduction and Motivation**

  Explore the (AdS/CFT) relation between Wilson loops and minimal area surfaces in AdS.

  Wilson loops $\rightarrow$ Strings $\rightarrow$ Minimal surfaces

- **Minimal area surfaces in hyperbolic space (EAdS$_3$)**

  Examples: Surfaces ending on a circle, and other analytical solutions.
● **Linear systems**

  Minimal area surfaces in flat space.
  Plateau problem and Douglas functional.
  Soap film problem.

● **Non-linear systems**

  The problem is studied in terms of a Schrödinger equation defined on the Wilson loop.

● **Conclusions**
\( \lambda \text{ small } \rightarrow \text{gauge th.} \)

\( \mathcal{N}=4 \text{ SYM} \)

\( \mathbb{S}^5: \quad X_1^2 + X_2^2 + \ldots + X_6^2 = 1 \)

\( \text{AdS}_5: \quad Y_1^2 + Y_2^2 + \ldots - Y_5^2 - Y_6^2 = -1 \)

\( \lambda \text{ large } \rightarrow \text{string th.} \)

Strings in \( \text{AdS}_5 \times \mathbb{S}^5 \)
Wilson loops: associated with a closed curve in space. Basic operators in gauge theories. E.g. $q\bar{q}$ potential.

Simplest example: single, flat, smooth, space-like curve (with constant scalar).

$$W = \frac{1}{N} \text{Tr} \, \hat{P} \exp \left\{ i \oint_C \left( A_\mu \frac{dx^\mu}{ds} + \theta_0^I \Phi_I \left| \frac{dx^\mu}{ds} \right| \right) ds \right\}$$
**String theory:** Wilson loops are computed by finding a minimal area surface *(Maldacena, Rey, Yee)*

Circle:

![Diagram of a circular surface](image)

*circular (~ Lobachevsky plane)*

Berenstein Corrado Fischler Maldacena Gross Ooguri, Erickson Semenoff Zarembo Drukker Gross, Pestun
Infinite parameter family of solutions:
Ishizeki, Ziama, M.K.

$$\lambda = i$$

$$\lambda = -\frac{1+i}{\sqrt{2}}$$

EAdS$_3$ Poincare coordinates
Also Euclidean surfaces in Minkowski signature:
(w/ A. Irrgang)

\textbf{AdS}_3 \text{ global coordinates}
What do we want to find?

Map $X(z, \bar{z}), Z(z, \bar{z})$ from unit circle to EAdS$_3$ ($\mathbb{H}_3$).

Minimizes area and has as boundary condition:

$|z| = 1 \Rightarrow Z(e^{i\theta}, e^{-i\theta}) = 0, \quad X(e^{i\theta}, e^{-i\theta}) = X(\theta(s))$
Consider first easier case: flat space

\[ z = \sigma + i\tau \]

\[ = r e^{i\theta} \]

\[ |z| = 1 \]

In conformal gauge:

\[ \{ x_i(s) \} \subset \mathbb{R}^n \]

\[ A = \frac{1}{2} \int \partial_a X_i \bar{\partial} a X_i \ d\sigma d\tau \]

\[ \partial \bar{\partial} X_i = 0 \quad \text{linear} \]

\[ X_i(r = 1, \theta) = x_i(\theta) \]
Using standard results for the Laplace equation, we immediately compute the area as (Douglas functional)

\[
\mathcal{A}[\theta(s)] = \frac{1}{16\pi} \int d\theta d\theta_0 \frac{(x_i(\theta) - x_i(\theta_0))^2}{\sin^2 \left(\frac{\theta - \theta_0}{2}\right)}
\]

However, we do not actually know \( \theta(s) \), namely we did not yet found the conformal coordinates.

It turns out that the correct \( \theta(s) \) is the one that minimizes the functional \( \mathcal{A}[\theta(s)] \).

Thus, the problem reduces to find the correct reparameterization \( \theta(s) \).
Minimal Area surfaces in EAdS$_3$

Equations of motion

\[ Z = \frac{1}{Y_0 - Y_3}, \quad X = \frac{Y_1 + iY_2}{Y_0 - Y_3} \]

\[ Y_0^2 - Y_1^2 - Y_2^2 - Y_3^2 = 1 \]

\[ \mathfrak{X} = \begin{pmatrix} Y_0 + Y_3 & Y_1 - iY_2 \\ Y_1 + iY_2 & Y_0 - Y_3 \end{pmatrix} = Y_0 + Y_i \sigma^i \]

\[ \partial \bar{\partial} \mathfrak{X} = \Lambda \mathfrak{X}, \quad \det \mathfrak{X} = 1, \quad \det(\partial \mathfrak{X}) = 0 = \det(\bar{\partial} \mathfrak{X}) \]

\[ ds^2 = 4e^{2\alpha} \, dz \, d\bar{z} \] Non-linear
Method of solution: Pohlmeyer reduction

Solve \( \partial \bar{\partial} \alpha = e^{2\alpha} + f(z) \bar{f}(\bar{z}) e^{-2\alpha} \)

And plug into

\[
J_z = \begin{pmatrix}
-\frac{1}{2} \partial \alpha & f(z) e^{-\alpha} \\
\lambda e^\alpha & \frac{1}{2} \partial \alpha
\end{pmatrix}
\quad \quad
J_{\bar{z}} = \begin{pmatrix}
\frac{1}{2} \bar{\partial} \alpha & \frac{1}{\lambda} e^\alpha \\
-\bar{f}(\bar{z}) e^{-\alpha} & -\frac{1}{2} \bar{\partial} \alpha
\end{pmatrix}
\]

\( dJ + J \wedge J = 0 \quad \partial J_{\bar{z}} - \bar{\partial} J_z + [J_z, J_{\bar{z}}] = 0 \)

Then solve the linear problem:

\[
\partial A = A J, \quad \bar{\partial} A = A \bar{J}.
\]

\[
X = A A A^\dagger
\]
**Integrability**

There are an infinite number of conserved charges. How can we use them?

\[ Q_\lambda = \text{Tr} \left[ \hat{P} e^{\int j_\lambda} \right] \]

Initial value problem

Boundary condition problem

\[ z = \sigma + i\tau = r e^{i\theta} \]

\[ Q_\lambda = 0, \quad \forall \lambda \]
\[ \langle 0 | \hat{W}_C | 0 \rangle \] The Wilson loop operator creates a multi-particle state/ boundary string state, we compute overlap with vacuum, zero charge component.

No radiation to infinity

String disappears.
No strings falling into the horizon
Minimal Area surfaces in $\text{EAdS}_3 (\mathbb{H}_3)$

Can we recast the problem into finding the correct reparameterization $\theta(s)$? Yes, $X(\theta)$ gives the necessary boundary conditions.

**Diagram:**
- $\alpha \to \infty$
- $|z| = 1$
- $z = \sigma + i\tau = re^{i\theta}$
- $\partial \bar{\partial} \alpha = e^{2\alpha} + f(z) \bar{f}(\bar{z}) e^{-2\alpha}$
- $ds^2 = 4 e^{2\alpha} dz d\bar{z}$
- $\beta_2(\theta) = \frac{1}{6} \left( \partial^2 \alpha - (\partial \alpha)^2 \right) |_{r \to 1}$
- $\xi = 1 - r^2$
- $\alpha \simeq -\ln \xi + \beta_2(\theta) \xi^2 + \beta_2(\theta) \xi^3 + \beta_4(\theta) \xi^4 + \cdots$
Boundary conditions in terms of boundary data $X(\theta)$. However, because of global conformal invariance there are many equivalent boundary data related by

$$\tilde{X}(\theta) = \frac{AX(\theta) + B}{CX(\theta) + D}$$

Use Schwarzian derivative!

$$\{f, \sigma\} = \frac{f'''(\sigma)}{f'(\sigma)} - \frac{3}{2} \left( \frac{f''(\sigma)}{f'(\sigma)} \right)^2$$

And indeed, the Schwarzian derivative gives the boundary data for $f(z)$ and $\alpha$.

$$V(\theta) = -\frac{1}{2} \{X(\theta), \theta\} = -\frac{1}{4} + 6\beta_2(\theta) - f(\theta)e^{2i\theta} + \bar{f}(\theta)e^{-2i\theta}$$
Derivation

The linear problem $d\mathbb{A} = \mathbb{A} \cdot j$ for the flat current $j$ can be written in the angular direction as $\partial_\theta (\psi_1 \psi_2) = (\psi_1 \psi_2) J^\theta$

Taking the limit $r \to 1$ and redefining $\chi(\theta) = \frac{\psi_1}{\sqrt{J_{21}^\theta}}$, we obtain

$$-\partial_\theta^2 \chi(\theta) + V(\theta, r = 1) \chi(\theta) = 0$$

With

$$V(\theta) = -\frac{1}{2} \{X(\theta), \theta\} = -\frac{1}{4} + 6\beta_2(\theta) - \lambda f(\theta)e^{2i\theta} + \frac{1}{\lambda} \bar{f}(\theta)e^{-2i\theta}$$

Reconstructing the boundary shape involves finding two l.i. solutions and computing the ratio:

$$X(\theta) = \frac{\chi_1(\theta)}{\tilde{\chi}_1(\theta)}$$

(Notice $\chi(\theta)$ is anti-periodic.)
Computation of the Area (assumption, $f$ no zeros)

The area is given by

$$A_f = -2\pi - 4 \int f \bar{f} e^{-2\alpha} d\sigma d\tau$$

From the cosh-Gordon eq. it follows that

$$j = j_z dz + j_{\bar{z}} d\bar{z}$$

$$j_z = -4f \sqrt{f} e^{-2\alpha}$$

$$j_{\bar{z}} = \frac{2}{\sqrt{f}} \left[ \bar{\partial}^2 \alpha - (\bar{\partial} \alpha)^2 \right]$$

$$dj = 0$$

$$\chi = \sqrt{f} \ dz$$

$$W(z) = \int_0^z \sqrt{f(z')}dz'$$
Then:

\[ A_f + 2\pi = -4 \int_D f \bar{f} e^{2\alpha} d\sigma d\tau = \frac{i}{2} \int_D j \wedge \bar{\chi} \]

\[ = \frac{i}{2} \int_D j \wedge d\bar{W} = \frac{i}{2} \int_D d(\bar{W} j) \]

\[ = -\frac{i}{2} \oint_{\partial D} \bar{W} (j_z dz + j_{\bar{z}} d\bar{z}) \]

Replacing the limiting behavior we get

\[ A_f = -2\pi \pm \frac{i}{2} \int \frac{\text{Re}\{X(s), s\} - \{w, s\}}{\partial_s \ln w} ds \]

\[ w(\theta) = i \int_{\theta}^{\theta} \sqrt{f(\theta') e^{2i\theta'}} d\theta' \]
In the case where \( f(z) \) has zeros (umbilical points)

\[
\int j \wedge \chi = \sum_i \left( \int_{a_i} j \int_{b_i} \chi - \int_{b_i} j \int_{a_i} \chi \right) \\
+ 2 \left( \int_d j \int_c \chi - \int_c j \int_d \chi \right) - \int_d j \int_d \chi \\
+ 2 \int_d W j
\]
Finding $\theta(s)$

Rewriting the Schrödinger equation using the coordinate $s$:

$$-\partial^2_s \chi(s) + V(\lambda, s) \chi(s) = 0$$

We find that the periodic potential is:

$$V(\lambda, s) = V_0(s) + \frac{i}{2} \left( \lambda + \frac{1}{\lambda} \right) V_1(s) + \frac{1}{2} \left( \lambda - \frac{1}{\lambda} \right) V_2(s)$$

$$V_0 + iV_1 = -\frac{1}{2} \{ X(s), s \} \quad \text{Known}$$

$$w(s) = \int^s \sqrt{V_2(s') + iV_1(s')} \, ds' , \quad V_2(s) + iV_1(s) = -2f(\theta)e^{2i\theta}(\partial_s \theta)^2$$
So, $V_0(s)$ and $V_1(s)$ are known but for $V_2(s)$ we need $\theta(s)$.

Alternatively, requiring that the solutions must be (anti)-periodic for all $\lambda$ allows, in principle, to compute $V_2(s)$ given $V_0(s)$ and $V_1(s)$. Not an easy problem, but it is equivalent to finding the area.

Moreover, when this is done, for each value of $\lambda$ such that $|\lambda|=1$ we have a different potential. The ratio of two solutions of the equation give a Wilson loop of different shape but the same expectation value. This implies the existence of a symmetry:

**$\lambda$-deformations**: symmetry not related to global symmetries of the theory but to integrability.
Important case: near circular Wilson loops

Near circular Wilson loops were studied initially by Semenoff and Young and subsequently by many authors (Drukker; Correa, Henn, Maldacena, Sever; Cagnazzo).

Recently Amit Dekel showed that the reparameterization $\theta(s)$ can be found order by order. He was able to compute expansion to order eight in some cases.

\[ X = e^{i\theta - \xi(\theta)} \]
New solutions \( f(z) = f_0 z^n \) w/ C. Huang and Yifei He

We propose a periodic potential

\[
V(\lambda, s) = -\frac{1}{4} + 6\beta_2 - \lambda f_0 e^{i(n+2)s} + f_0 \frac{1}{\lambda} e^{-i(n+2)s}
\]

such that all the solutions of the corresponding Schrödinger equation are anti-periodic. Indeed, setting \( \lambda = e^{i\phi} \), changing \( \lambda \) only shifts \( s \) preserving the periodicity. Moreover, the change of variables

\[
u(s) = \frac{(n+2)s + \phi}{2} + \frac{\pi}{4}
\]

reduces the Schrödinger equation to the Mathieu equation:
We can find Floquet solutions $\chi_\nu(u + \pi) = e^{\pi i \nu} \chi_\nu(u)$.

$\chi_\nu(u) = e^{i \nu u} p_\nu(q, u)$

We need that happens for a discrete set of eigenvalues $\nu = \frac{2k + 1}{n + 2}$, namely

$\beta_{2,k} = \frac{1}{24} \left[ 1 - (n + 2)^2 a_k(q) \right]$
The shape of the contour is given by the ratio of two linearly independent solutions

\[ X(u) = \frac{\chi_\nu(u)}{\chi_\nu(-u)} = e^{2i\nu u} \frac{p_\nu(q, u)}{p_\nu(q, -u)} \]

And the area by (using the same reasoning than before)

\[ A_f = -2\pi + \frac{\pi}{n + 2} - (n + 2)\pi a_k(q) \]
Example:

\[ n = 2 \]

\[ A = -6.660397 \]

Analytical

Numerical
New solutions \( f(z) = f_0 z^n \) Minkowski case

\[
\begin{align*}
  x_+(u) &= \frac{\chi_\nu(u)}{\chi_\nu(-u)} = e^{2i\nu u} \frac{p_\nu(q, u)}{p_\nu(q, -u)} \\
  x_-(u) &= \frac{\chi_\nu(u + \frac{\pi}{2})}{\chi_\nu(-u + \frac{\pi}{2})} = e^{2i\nu u} \frac{p_\nu(q, u + \frac{\pi}{2})}{p_\nu(q, -u + \frac{\pi}{2})}
\end{align*}
\]

\[a = \frac{1 - 24\beta_2}{(n + 2)^2}, \quad q = \frac{4f_0}{(n + 2)^2}\]

For \( n=0 \) it is the solution proposed by J. Toledo
For \( f_0 \to \infty \) it is a regular polygon (Alday-Maldacena).
Example:

\[ n=2 \]

\[ n=4 \]
For $f_0=0$ it is a circle. The Mathieu equation reduces to the free particle equation ($q=0$). When $f_0$ is small then $q$ is small and one can use perturbation theory to compute the eigenvalues and from there the area. Alternatively one can use Dekel’s procedure. The results agree.

\[
A_{reg} = -2\pi - \frac{\pi q^2}{2\nu (\nu^2 - 1)} - \frac{(5\nu^2 + 7)\pi q^4}{32\nu (\nu^2 - 4)(\nu^2 - 1)^3} + O(q^5)
\]
Conclusions

Wilson loops in terms of minimal area surfaces:

The Wilson loop can be computed if we know the reparameterization \( \theta(s) \). We have to study a certain Schrödinger equation with potential given by the shape.

The reparameterization \( \theta(s) \) can be found by imposing the vanishing of an infinite number of conserved charges. (The Wilson loop creates a string with the quantum numbers of the vacuum)

The system has a symmetry, \( \lambda \)-deformations that is not related to the global symmetries of the theory but to its integrability properties.
Future work

Relation to Y-sytem approach (J. Toledo).

Extend to higher dimensional spaces (AdS\(_n\)).
   General integrable system

Compute quantum corrections.

We need to explore the field theory side.