Geometric invariants associated with linear transformations

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Outline

History:
Geometric phase (…Pancharatnam, Mead and Truhlar, Hannay, Berry, Simon, Wilczek and Zee …)

Tools:
Algebraic and analytic methods
Differential forms (phases are fluxes of forms)
Resolvent (a detector of degeneracies)

Goal:
Classify geometric invariants associated with linear transformations
Study spectrum degeneracies (direct and inverse problems)
Geometric phase

A Hamiltonian $H(x(t))$ depends on a parameter $x(t)$ that varies in time.

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi$$

$$H\phi = E\phi$$

Adiabatic evolution of $\phi(t)$:

$$\psi(t) = \exp \left[ -\frac{i}{\hbar} \int_0^t E(x(t'))dt' \right] e^{i\gamma(t)} \phi(x(t))$$

$$\frac{d\gamma}{dt} = i \left( \phi, \frac{d\phi}{dt} \right) = i \left( \phi, \nabla_x \phi \cdot \frac{dx}{dt} \right)$$
Geometric phase

Holonomy

$$\gamma = \int_C \omega = \int_S \Omega, \quad C = \partial S$$

Connection $$\omega = i(\varphi, d\varphi)$$
Curvature $$\Omega = d\omega = i(d\varphi, d\varphi)$$

How can $\gamma$ be generalized?
Algebraic method
Set-up

Let $D \subset \mathbb{R}^d$ be a region.
Consider a linear transformation $T(x): \mathbb{C}^n \rightarrow \mathbb{C}^n$.
Let $\sigma(x)$ be the spectrum of $T(x)$.
Let $\lambda(x) \in \sigma(x)$ be an eigenvalue of $T(x)$ with the algebraic multiplicity $m(x)$.
The function $m$ is constant in $D$ except for the points of degeneracy of $T$ where it changes discontinuously by an integer.
Let $\lambda_1(x), \ldots, \lambda_r(x), r(x) \leq n$ be distinct eigenvalues of $T(x)$.

The geometric eigenspace for $\lambda_j(x)$ is

$$S_j(x) = \{ \text{span } u_j(x) : T(x)u_j(x) = \lambda_j(x)u_j(x) \}.$$ 

The eigenprojection is $P_j(x) : \mathbb{C}^n \to S_j(x)$.

The eigennilpotent is

$$N_j(x) = (T(x) - \lambda_j(x)I)P_j(x) = P_j(x)(T(x) - \lambda_j(x)I).$$

The spectral representation of $T(x)$:

$$T(x) = \sum_{j=1}^{r(x)} \lambda_j(x)P_j(x) + \sum_{j=1}^{r(x)} N_j(x).$$

Algebraic properties:

$$P_j^k(x) = P_j(x), \ k \geq 1,$$

$$N_j^k(x) = 0, \ k \geq m_j(x),$$

$$P_j(x)N_j(x) = N_j(x)P_j(x) = N_j(x).$$
Invariants

From now on, all equations are for the same eigenvalue \( j \) and \( x \). Global changes of bases are related by \( x \)-independent similarity transformations \( U \):

\[
T' = UTU^{-1}, \quad \lambda' = \lambda, \quad P' = UPU^{-1}, \quad N'(x) = UNU^{-1},
\]

\[
d\lambda' = d\lambda, \quad dP' = UdPU^{-1}, \quad dN' = UdNU^{-1}.
\]

We look for invariants under similarity transformations. The most general monomial differential \( r \)-form constructed from \( P \), \( dP \), \( N \), \( dN \) that is invariant under similarity transformations is

\[
Q_{k_1\ldots k_r} = \text{tr} \left( X_{k_1} dY_{k_1} \wedge \cdots \wedge X_{k_r} dY_{k_r} \right),
\]

where \( Y_{k_p} \in \{P, N\} \) and \( X_{k_p} \) is a product of any number of operators \( P \) and \( N \).

Use algebraic properties of \( P \) and \( N \) to deduce

\[
X_{k_p} \in \{I, P, N, \ldots, N^{m-1}\}.
\]
Invariants

Identities $PdPP = PdPN = NdPP = NdPN = 0$ imply

$$X_{k_p}dY_{k_p} \in \{dP, PdP, N^{l_p}dP, dN, PdN, N^{l_p}dN\}, \quad 1 \leq l_p \leq m - 1.$$  

We now have a combinatorial problem of finding all independent non-zero forms

$$Q_{k_1 \ldots k_r} = \text{tr} \left( X_{k_1}dY_{k_1} \land \cdots \land X_{k_r}dY_{k_r} \right).$$

The solution depends on the degree $m$ of the nilpotent operator $N$ (which is also the algebraic multiplicity of the eigenvalue $\lambda$).
Invariants for $m = 1$

The eigenvalue $\lambda$ is simple, hence $N = 0$. If $m = 1$ for all eigenvalues of $T$, then $T$ is diagonalizable.

There are 2 independent 1-forms: $dP$ and $PdP$. $P^2 = P$ gives $dPP + PdP = dP$, which implies that the only non-zero invariant forms are $Q_{2s} = \text{tr } \Phi^s$, where $\Phi = PdP \wedge dPP$.

The only invariant forms of odd order are $\text{tr } (\Phi^s \wedge PdP)$ and $\text{tr } (\Phi^s \wedge dPP)$. Both vanish as a result of the identity $dPP + PdP = dP$ and the cyclic property of trace.
1-form invariants for $m = 2$

The eigenvalue $\lambda$ is degenerate, $N \neq 0$, $N^2 = 0$. The operator $T$ is not diagonalizable.

There are 6 independent 1-forms:
\[
\text{tr } dP, \text{ tr } PdP, \text{ tr } NdP, \text{ tr } dN, \text{ tr } PdN, \text{ tr } NdN
\]

Zero forms
Nonzero forms

Identities:
\[
\text{tr } dP = 0, \text{ tr } PdP = 0, \text{ tr } dN = 0, \text{ tr } NdN = 0, \text{ tr } (NdP + PdN) = 0.
\]

The only independent nonzero 1-form is $Q_1 = \text{tr } (NdP) = -\text{tr } (PdN)$.

It is not clear what the simplest example of $T$ should be for $Q_1 \neq 0$. 
2-form invariants for $m = 2$

There are 21 possible 2-forms constructed from 6 forms $dP, PdP, NdP, dN, PdN, NdN$:

\[
\begin{align*}
\text{tr} (dP \wedge dP) & \quad \text{tr} (PdP \wedge NdP) & \quad \text{tr} (NdP \wedge NdN) \\
\text{tr} (dP \wedge PdP) & \quad \text{tr} (PdP \wedge dN) & \quad \text{tr} (dN \wedge dN) \\
\text{tr} (dP \wedge NdP) & \quad \text{tr} (PdP \wedge PdN) & \quad \text{tr} (dN \wedge PdN) \\
\text{tr} (dP \wedge dN) & \quad \text{tr} (PdP \wedge NdN) & \quad \text{tr} (dN \wedge NdN) \\
\text{tr} (dP \wedge PdN) & \quad \text{tr} (NdP \wedge NdP) & \quad \text{tr} (PdN \wedge PdN) \\
\text{tr} (dP \wedge NdN) & \quad \text{tr} (NdP \wedge dN) & \quad \text{tr} (PdN \wedge NdN) \\
\text{tr} (PdP \wedge PdP) & \quad \text{tr} (NdP \wedge PdN) & \quad \text{tr} (NdN \wedge NdN)
\end{align*}
\]

Zero forms by the trace property $\text{tr} (A \wedge B) = -\text{tr} (B \wedge A)$

Zero forms by the identities $PdPP = PdPN = NdPP = NdPN = 0$

Nonzero forms
### 2-form invariants for $m = 2$

\[
\begin{align*}
    \text{tr} (dP \wedge PdP) &= \text{tr} (dP \wedge NdN) &= \text{tr} (dN \wedge PdN) \\
    \text{tr} (dP \wedge NdP) &= \text{tr} (PdP \wedge dN) &= \text{tr} (dN \wedge NdN) \\
    \text{tr} (dP \wedge dN) &= \text{tr} (NdP \wedge dN) &= \text{tr} (PdN \wedge NdN) \\
    \text{tr} (dP \wedge PdN) &= \text{tr} (dP \wedge NdP) &= \text{tr} (NdP \wedge dN)
\end{align*}
\]

### Zero forms

A group of related nonzero forms

A group of related nonzero forms

### Further identities:

\[
\begin{align*}
    \text{tr} (dP \wedge dN) &= 0 \\
    \text{tr} (dP \wedge NdP) &= \text{tr} (dP \wedge PdN) &= -\text{tr} (PdP \wedge dN) \\
    \text{tr} (dP \wedge NdN) &= \text{tr} (dN \wedge PdN) &= -\text{tr} (NdP \wedge dN) \\
    \text{tr} (PdN \wedge NdN) &= 0 \\
    \text{tr} (dN \wedge NdN) &= 0
\end{align*}
\]
**2-form invariants for $m = 2$**

The only independent nonzero 2-forms $Q_2$ are

\[ Q_2^{(1)} = \text{tr} \left( dP \wedge PdP \right), \]

\[ Q_2^{(2)} = \text{tr} \left( dP \wedge NdP \right) = \text{tr} \left( dP \wedge PdN \right) = -\text{tr} \left( PdP \wedge dN \right), \]

\[ Q_2^{(3)} = \text{tr} \left( dP \wedge NdN \right) = \text{tr} \left( dN \wedge PdN \right) = -\text{tr} \left( NdP \wedge dN \right). \]

It is easy to give examples of $T$ for $Q_2^{(1)} \neq 0$.

It is not clear what the simplest example of $T$ should be for $Q_2^{(2)} \neq 0$ or $Q_2^{(3)} \neq 0$. 
Geometric quantities
Geometric quantities $\omega$, $\Omega$

Let $\{v_1, \ldots, v_{m'}\}$ be an orthonormal basis of the eigenspace $S$ corresponding to $\lambda$.

**geometric multiplicity** $m' \leq$ **algebraic multiplicity** $m$

$m' = m$ for semisimple eigenvalues

Consider the $n \times m'$ matrix $V$ whose columns are $v_1, \ldots, v_{m'}$.

$$V^* V = I,$$

$$VV^* = P.$$

Introduce the $m' \times m'$ matrices of

the connection 1-form $\omega = V^* dV$,

the curvature 2-form $\Omega = d\omega + \omega \wedge \omega$. 
Relations between $\Phi$, $\Psi$, $\Omega$

For any integer $s \geq 0$:

$$\Phi^s = V \Omega^s V^*,$$

$$\Psi^{(s+1)} = dPV \wedge \Omega^s \wedge V^*dP.$$

$\Phi^s$ and $\Omega^s$ are unitarily equivalent.

**Proof:** For $s \geq 1$:

$$\Phi^s = \Phi \wedge \Phi^{(s-1)} = PdP \wedge dPP \wedge \Phi^{(s-1)} = PdP \wedge dPP \wedge \Phi^{(s-1)}P$$

$$= VV^*dP \wedge dP \wedge \Phi^{(s-1)}VV^*, $$

where we used $P\Phi = \Phi P = \Phi$, which follows from $\Phi = PdP \wedge dPP$.

We first prove

$$V^*dP \wedge dP \wedge \Phi^{(s-1)}V = \Omega^s$$

by induction.
Relations between $\Phi$, $\Psi$, $\Omega$

$s = 1$:

\[ V^* dP \wedge dPV = V^* (dVV^* + VdV^*) \wedge (dVV^* + VdV^*)V \]
\[ = V^* dV \wedge V^* dV + V^* dV \wedge dV^* V \]
\[ + dV^* \wedge dV + dV^* V \wedge dV^* V \]
\[ = \omega \wedge \omega - \omega \wedge \omega + d\omega + \omega \wedge \omega \]
\[ = \Omega. \]

The inductive step:

\[ V^* dP \wedge dP \wedge \Phi^{\wedge s} V = V^* dP \wedge dP \wedge \Phi \wedge \Phi^{\wedge (s-1)} V \]
\[ = V^* dP \wedge dP \wedge PdP \wedge dPP \wedge \Phi^{\wedge (s-1)} V \]
\[ = V^* dP \wedge dP \wedge VV^* dP \wedge dP \wedge \Phi^{\wedge (s-1)} V \]
\[ = V^* dP \wedge dPV \wedge \Omega^{\wedge s} \]
\[ = \Omega^{\wedge (s+1)}. \]
Relations between $\Phi$, $\Psi$, $\Omega$

This completes the proof of

$$V^*dP \land dP \land \Phi^{(s-1)}V = \Omega^s$$

and consequently of

$$\Phi^s = V\Omega^s V^*,$$

The proof of

$$\Psi^{(s+1)} = dPV \land \Omega^s \land V^*dP$$

is easy by using $\Psi^{(s+1)} = dPP \land \Phi^s \land PdP$, which follows from the definitions $\Phi = PdP \land dPP$ and $\Psi = dPP \land PdP$:

$$\Psi^{(s+1)} = dPP \land V\Omega^s V^* \land PdP = dPV \land \Omega^s \land V^*dP.$$
Invariants from $\Phi$, $\Psi$, $\Omega$

Change a basis in $\mathbb{C}^n$ with the $x$-independent $n \times n$ matrix of a unitary transformation $U$:

$$
V \mapsto UV, \quad V^* \mapsto V^*U^*, \quad P \mapsto UPU^*, \quad dP \mapsto UdPU^*,
$$

$$
\omega \mapsto \omega, \quad \Omega \mapsto \Omega, \quad \Phi \mapsto U\Phi U^*, \quad \Psi \mapsto U\Psi U^*, \quad Q \mapsto Q.
$$

The only invariant differential forms:

$$
\text{tr} (\Phi^s) = \text{tr} (\Omega^s),
$$

$$
\text{tr} (\Psi^s) = (-1)^{2s-1} \text{tr} (\Omega^s),
$$

$$
\text{tr} (\Phi^s \wedge PdP) = \text{tr} (PdP \wedge \Psi^s) = 0,
$$

$$
\text{tr} (\Psi^s \wedge dPP) = \text{tr} (dPP \wedge \Phi^s) = 0.
$$

The only independent invariant differential forms are nonzero Chern classes:

$$
\text{tr} (\Omega^s), \quad 0 \leq s \leq [d/2],
$$

$[d/2]$ is the integer part of $d/2$. 
Invariants from $\Phi$, $\Psi$, $\Omega$

$x$-dependent $m' \times m'$ matrix of a unitary gauge transformation $W$:

$$
\omega \mapsto WV^* d(VW^*) = W^* \omega W + W^* dW,
\Omega \mapsto d(W^* \omega W + W^* dW) + (W^* \omega W + W^* dW) \wedge (W^* \omega W + W^* dW) = W^* \Omega W
$$

$V \mapsto VW^*$, $V^* \mapsto W^* V^*$, $P \mapsto P$, $dP \mapsto dP$,
$$
\omega \mapsto W^* \omega W + W^* dW, \quad \Omega \mapsto W^* \Omega W, \quad \Phi \mapsto \Phi, \quad \Psi \mapsto \Psi, \quad Q \mapsto Q.
$$

$x$-independent $n \times n$ matrix of a unitary transformation $U$:

$$
V \mapsto UV, \quad V^* \mapsto V^* U^*, \quad P \mapsto UPU^*, \quad dP \mapsto UdPU^*,
\omega \mapsto \omega, \quad \Omega \mapsto \Omega, \quad \Phi \mapsto U\Phi U^*, \quad \Psi \mapsto U\Psi U^*, \quad Q \mapsto Q.
$$

$Q$ is invariant under both $U$ and $W$ transformations.
Invariants from $\Phi, \Psi, \Omega$

Perhaps it’s useful to collect all Chern classes into the Chern form as the coefficients of the characteristic polynomial of the curvature form $\Omega$:

$$\det \left( I + \frac{iz}{2\pi} \Omega \right) = \text{tr} I + \frac{iz}{2\pi} \text{tr} \Omega + \frac{z^2}{8\pi^2} \left[ \text{tr} \Omega^2 - (\text{tr} \Omega)^2 \right] + \frac{iz^3}{48\pi^3} \left[ -2\text{tr} \Omega^3 + 3\text{tr} \Omega^2 \text{tr} \Omega - \text{tr} \Omega^3 \right] + \cdots .$$
**Behavior near points of degeneracy**

Suppose $T(x_*) = 0$ for some $x_* \in D$. 
$\lambda(x_*) = 0$, $m(x_*) = m'(x_*) = n$, $P(x)$ is singular at $x = x_*$.  
The eigenvalue’s degeneracy and the eigenprojection’s singularity are typically lifted for $x$ near $x_*$. 

How do these functions behave near the point of degeneracy?  
No loss of generality for $x_* = 0$.  
$T$ must be a homogeneous function of degree $q$ near $x = 0$, 
$T(tx) = t^q T(x)$.  
Rescale $x = y^{1/q}$ to get the degree 1, $T((ty)^{1/q}) = tT(y^{1/q})$.  

**Homogeneous functions of $x$:**  
$R(\zeta, x)$ (degree $-1$)  
$\lambda(x)$ (degree 1)  
$P(x), dP(x), N(x), dN(x), \Phi(x), \Psi(x), \omega(x), \Omega(x), Q(x)$ (degree 0)
Behavior near points of degeneracy

Let $M \subset D$ be a $2s$-dimensional region. Consider a family of $2s$-dimensional regions parametrized by $0 < t \leq 1$,

$$M_t = \{ x \in D: x = ty, y \in M \}$$

such that $M_t \subset M_{t'}$ if $t < t'$.

It follows that the flux through $M_t$ does not depend on $t$:

$$\int_{x \in M_t} Q(x) = \int_{y \in M} Q(ty) = \int_{y \in M} Q(y).$$
From singularities to degeneracies

Suppose \( \lim_{t \to 0} \int_{M_t} Q \not\in \{0, -\infty, \infty\} \).

From

\[ Q = \text{tr} \, \Phi^s, \]
\[ \Phi = PdP \wedge dPP = \frac{1}{2} \sum_{ij} \Phi_{ij} dx_i \wedge dx_j \]

it follows that
\( \Phi_{ij} \) must be singular at \( x = 0 \).

\( P \) cannot be singular because \( P^2 = P \).

Thus \( \partial P / \partial x_i \) must be singular.
From singularities to degeneracies

$P$ is nonsingular at $x = 0$.
$\partial P/\partial x_i$ is singular at $x = 0$.

There must be an abrupt change of the eigenspace $S(x)$ at $x = 0$. It follows that the multiplicity $m(x)$ has a discontinuous jump at $x = 0$.

$$\lim_{t \to 0} \int_{M_t} Q \notin \{0, -\infty, \infty\} \text{ implies}$$

$$Q = O(t^0), \Phi = O(t^0), \Psi = O(t^0), \Omega = O(t^0), P = O(t^0), t \to 0.$$ 

As a result, $Q, \Phi, \Psi, \Omega, P$ are homogeneous functions of degree 0.
Example: $n = 2, m = 1, m' = 1$

$$T(x) = \begin{bmatrix} x_4 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_4 - x_3 \end{bmatrix}, \ x_1, x_2, x_3, x_4 \in \mathbb{R}$$

$$\lambda_{\pm}(x) = x_4 \pm r, \quad r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$$

$$\lambda_+ \neq \lambda_-, \ r \neq 0$$

$$R(\zeta, x) = ((\zeta - x_4)^2 - r^2)^{-1} \begin{bmatrix} \zeta - x_4 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & \zeta - x_4 - x_3 \end{bmatrix}$$
Example: \( n = 2, m = 1, m' = 1 \)

\[
R^{(k)}_\pm (x) = 0, \ k \leq -2,
\]

\[
R^{(-1)}_\pm (x) = \pm \frac{1}{2r} \begin{bmatrix} x_3 \pm r & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \pm r \end{bmatrix},
\]

\[
R^{(k)}_\pm (x) = -\frac{(\mp 1)^k}{(2r)^{k+2}} \begin{bmatrix} x_3 \mp r & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \mp r \end{bmatrix}, \ k \geq 0.
\]

\[
P_\pm (x) = \pm \frac{1}{2r} \begin{bmatrix} x_3 \pm r & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \pm r \end{bmatrix}, \ N_\pm (x) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\]

\[
A_\pm (x) = -\frac{1}{(2r)^2} \begin{bmatrix} x_3 \mp r & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \mp r \end{bmatrix}
\]
Example: $n = 2, m = 1, m' = 1$

$$
\Phi_{\pm}(x) = \frac{i}{4r^4} \left[ \begin{array}{ccc} x_3 \pm r & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \pm r \end{array} \right] \sigma(x),
$$

$$
\sigma(x) = x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_3 dx_1 \wedge dx_2.
$$

$$
Q^{(1)}_{2,\pm} = \text{tr} \ \Phi_{\pm}(x) = \pm \frac{i}{2r^3} \sigma(x)
$$

$$
V_{\pm}(x) = \frac{e^{i\varphi_{\pm}(x)}}{(2r(r \pm x_3))^{1/2}} \left[ \begin{array}{c} x_3 \pm r \\ x_1 + ix_2 \end{array} \right], \text{ where } \varphi_{\pm}: D \to \mathbb{R} \text{ are arbitrary}
$$

$$
\omega_{\pm}(x) = \frac{i(x_1 dx_2 - x_2 dx_1)}{2r(r \pm x_3)} + i\varphi_{\pm}(x)
$$

$$
\Omega_{\pm}(x) = \pm \frac{i}{2r^3} \sigma(x)
$$

$P, N, \Phi, \omega, \Omega, Q^{(1)}_\omega$ are homogeneous functions of degree 0.
Example: \( n = 2, m = 2, m' = 1 \)

\[
T(x) = \begin{bmatrix}
    x_4 + x_3 & x_1 - ix_2 \\
    x_1 + ix_2 & x_4 - x_3
\end{bmatrix}, \quad x_1, x_2, x_3, x_4 \in \mathbb{C}
\]

\[
\lambda_\pm(x) = x_4 \pm r, \quad r = (x_1^2 + x_2^2 + x_3^2)^{1/2}
\]

\[
\lambda_+ = \lambda_-, \quad r = 0
\]

\[
R(\zeta, x) = (\zeta - x_4)^{-2} \begin{bmatrix}
    \zeta - x_4 + x_3 & x_1 - ix_2 \\
    x_1 + ix_2 & \zeta - x_4 - x_3
\end{bmatrix}
\]
Example: $n = 2$, $m = 2$, $m' = 1$

$$R_{\pm}^{(k)}(x) = 0, \ k \leq -3,$$

$$R_{\pm}^{(-2)}(x) = \begin{bmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{bmatrix},$$

$$R_{\pm}^{(-1)}(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$R_{\pm}^{(k)}(x) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \ k \geq 0.$$
Example: $n = 2, m = 2, m' = 1$

The only invariant 2-form is

$$N_\pm(x)dN_\pm(x) \wedge dN_\pm(x)N_\pm(x) = 2iN_\pm(x)\sigma(x)$$

$$= \begin{bmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{bmatrix} \sigma(x).$$

$$\text{tr} \left( N_\pm(x)dN_\pm(x) \wedge dN_\pm(x)N_\pm(x) \right) = 0$$

Once again, all invariant 2-form are zero.

Nothing more to be done for the case $n = 2$.

Preliminary results for $n = 3, m = 2$ are also negative: there are no nonzero invariant 2-form.

Of course, no problem for $m = 1$, as $Q^{(1)}_2 = \text{tr} \Omega \neq 0$. 
Analytic method
Resolvent

Let $\rho(x) = \mathbb{C}\setminus \sigma(x)$ be the resolvent set of $T(x)$. The resolvent of $T(x)$ is $R(\zeta, x) = (\zeta I - T(x))^{-1}$, where $\zeta \in \rho(x)$. $R(\zeta, x)$ is a meromorphic function of $\zeta$. The eigenvalues of $T(x)$ are isolated singularities of $R(\zeta, x)$ in $\zeta$.

$R(\zeta, x)$ can be expanded in the Laurent series about $\lambda_j(x)$:

$$R(\zeta, x) = \sum_{k=-\infty}^{\infty} R_j^{(k)}(x)(\zeta - \lambda_j(x))^k,$$

$$R_j^{(k)}(x) = \frac{1}{2\pi i} \int_{C_j(x)} \frac{R(\zeta, x)d\zeta}{(\zeta - \lambda_j(x))^{k+1}}.$$

$C_j(x)$ is a positively-oriented simple contour in $\mathbb{C}$ enclosing the point $\zeta = \lambda_j(x)$ and excluding all other eigenvalues of $T(x)$. 
Resolvent

From $R(\zeta, x) = (\zeta I - T(x))^{-1}$ we find
$$R^{-1}(\zeta, x) - R^{-1}(\zeta', x) = (\zeta - \zeta')I,$$
which leads to the resolvent identity
$$R(\zeta, x) - R(\zeta', x) = (\zeta' - \zeta)R(\zeta, x)R(\zeta', x).$$

Use this identity to reduce any product $R(\zeta_1, x) \cdots R(\zeta_r, x)$ to a linear combination of $R(\zeta_1, x), \ldots, R(\zeta_r, x)$ when all $\zeta_1, \ldots, \zeta_r$ are distinct.

Expand the resolvents in the Laurent series and find
$$R_j^{(k)}(x)R_j^{(l)}(x) = \begin{cases} 
-R_j^{(k+l+1)}(x), & k \geq 0, \ l \geq 0, \\
R_j^{(k+l+1)}(x), & k < 0, \ l < 0, \\
0, & \text{otherwise,} 
\end{cases}$$
$$T(x)R_j^{(k)}(x) = R_j^{(k)}(x)T(x) = R_j^{(k-1)}(x) + \lambda_j(x)R_j^{(k)}(x) - I \delta_{k,0}. $$
Resolvent

\[ R_j^{(k)}(x) = N_j^{-k-1}(x), \quad k \leq -2, \]
\[ R_j^{(-1)}(x) = P_j(x), \]
\[ R_j^{(k)}(x) = (-1)^k A_j^{k+1}(x), \quad k \geq 0, \]

where

\[ P_j^2(x) = P_j(x), \]
\[ N_j(x)P_j(x) = P_j(x)N_j(x) = N_j(x), \]
\[ A_j(x)P_j(x) = P_j(x)A_j(x) = 0, \]
\[ T(x)P_j(x) = P_j(x)T(x) = \lambda_j(x)P_j(x) + N_j(x), \]
\[ (\lambda_j(x)I - T(x))A_j(x) = A_j(x)(\lambda_j(x)I - T(x)) = I - P_j(x). \]
Resolvent

\( N_j(x) \) is nilpotent with degree \( m_j(x) \leq n \), so that \( N_j(x)^{m_j(x)} = 0 \). The Laurent series about \( \lambda_j(x) \) becomes

\[
R(\zeta, x) = \sum_{k=1}^{m_j(x)-1} \frac{N_j^k(x)}{(\zeta - \lambda_j(x))^{k+1}} + \frac{P_j(x)}{\zeta - \lambda_j(x)} + \sum_{k=0}^{\infty} (-1)^k A_j^{k+1}(x)(\zeta - \lambda_j(x))^k.
\]

From the Cauchy integral formula:

\[
N_j(x) = \frac{1}{2\pi i} \int_{C_j(x)} R(\zeta, x)(\zeta - \lambda_j(x)) d\zeta,
\]
\[
P_j(x) = \frac{1}{2\pi i} \int_{C_j(x)} R(\zeta, x) d\zeta,
\]
\[
A_j(x) = \frac{1}{2\pi i} \int_{C_j(x)} R(\zeta, x)(\zeta - \lambda_j(x))^{-1} d\zeta.
\]
Assume that $T(x)$ is a smooth function of $x \in D$, which implies that $R(\zeta, x)$ is a smooth function of $\zeta \in \rho(x)$ and $x \in D$.

Consider the differential 1-forms

$$d_\zeta R(\zeta, x) = \frac{\partial R(\zeta, x)}{\partial \zeta} d\zeta = -R^2(\zeta, x) d\zeta,$$

$$d_x R(\zeta, x) = \sum_{a=1}^{d} \frac{\partial R(\zeta, x)}{\partial x_a} dx_a = \sum_{a=1}^{d} R(\zeta, x) \frac{\partial T(x)}{\partial x_a} R(\zeta, x) dx_a,$$

$$dR(\zeta, x) = d_\zeta R(\zeta, x) + d_x R(\zeta, x),$$

$$d_x R^{(k)}_j (x) = \sum_{a=1}^{d} \frac{\partial R^{(k)}_j (x)}{\partial x_a} dx_a.$$
Differential forms from the resolvent

To compute

$$d_x R_j^{(k)}(x) = \frac{1}{2\pi i} d_x \int_{C_j(x)} \frac{R(\zeta, x) d\zeta}{(\zeta - \lambda_j(x))^{k+1}},$$

we need to differentiate the contour and integrand with respect to $x$. Take for $C_j(x)$ a circle of (sufficiently small) radius $r$ centered at $\zeta = \lambda_j(x)$ and find

$$d_x R_j^{(k)}(x) = \frac{1}{2\pi i} \int_0^{2\pi} d_x R(\lambda_j(x) + re^{i\phi}, x) \frac{ire^{i\phi} d\phi}{(re^{i\phi})^{k+1}}$$

$$= \frac{1}{2\pi i} \int_0^{2\pi} \left[ \frac{\partial R(\zeta, x)}{\partial \zeta} d_x \lambda_j(x) + d_x R(\zeta, x) \right]_{\zeta = \lambda_j(x) + re^{i\phi}} \frac{ire^{i\phi} d\phi}{(re^{i\phi})^{k+1}}$$

$$= \frac{1}{2\pi i} \int_{C_j(x)} R(\zeta, x) d_x \left( T(x) - I\lambda_j(x) \right) R(\zeta, x) \frac{d\zeta}{(\zeta - \lambda_j(x))^{k+1}}.$$
Differential forms from the resolvent

Use $R(\zeta, x) = \sum_{k=-\infty}^{\infty} R_j^{(k)}(x)(\zeta - \lambda_j(x))^k$ to find

$$d_x R_j^{(k)}(x) = \frac{1}{2\pi i} \sum_{l=-\infty}^{\infty} \sum_{l'=-\infty}^{\infty} R_j^{(l)}(x) d_x \left( T(x) - I\lambda_j(x) \right) R_j^{(l')}(x)$$

$$\times \int_{C_j(x)} (\zeta - \lambda_j(x))^{-k-1+l+l'} d\zeta$$

$$= \sum_{l=-\infty}^{\infty} R_j^{(l)}(x) d_x \left( T(x) - I\lambda_j(x) \right) R_j^{(k-l)}(x).$$

The lowest power in the Laurent series is $-m_j(x)$, hence

$$d_x R_j^{(k)}(x) = \sum_{l=-m_j(x)}^{k+m_j(x)} R_j^{(l)}(x) d_x \left( T(x) - I\lambda_j(x) \right) R_j^{(k-l)}(x).$$
Differential forms from the resolvent

Take products of \( R(\zeta_1, x), R(\zeta_2, x), \ldots \) and \( d_x R(\zeta_1, x), d_x R(\zeta_2, x), \ldots \) and define differential forms of various orders. Some of these forms are dependent on the others. Use the resolvent identity \( R(\zeta, x) - R(\zeta', x) = (\zeta' - \zeta) R(\zeta, x) R(\zeta', x) \) to find some of these dependencies. Take \( d_x \):

\[
d_x R(\zeta, x) - d_x R(\zeta', x) = (\zeta' - \zeta) \left( d_x R(\zeta, x) R(\zeta', x) + R(\zeta, x) d_x R(\zeta', x) \right).
\]

Multiplying this by \( R(\zeta'', x) \) either on the left or on the right:

\[
(\zeta' - \zeta)(\zeta'' - \zeta') R(\zeta, x) d_x R(\zeta', x) R(\zeta'', x) = (\zeta' - \zeta'') d_x R(\zeta', x) R(\zeta'', x)
\]

\[
+ (\zeta' - \zeta) d_x R(\zeta, x) R(\zeta', x) + (\zeta'' - \zeta) d_x R(\zeta, x) R(\zeta'', x),
\]

\[
(\zeta - \zeta'')(\zeta' - \zeta) R(\zeta'', x) d_x R(\zeta, x) R(\zeta', x) = (\zeta - \zeta') d_x R(\zeta, x) R(\zeta', x)
\]

\[
+ (\zeta'' - \zeta) d_x R(\zeta'', x) R(\zeta, x) + (\zeta' - \zeta'') d_x R(\zeta'', x) R(\zeta', x).
\]
Differential forms from the resolvent

Apply these identities iteratively and express any quantity of the type

\[ R(\zeta_1, x) \cdots R(\zeta_k, x)d_x R(\zeta_{k+1}, x) \]
\[ \wedge R(\zeta_{k+2}, x) \cdots R(\zeta_{k+l+1}, x)d_x R(\zeta_{k+l+2}, x) \wedge \cdots , \]

where \( \zeta_1, \zeta_2, \ldots \) are all distinct, as a linear combination (with coefficients depending only on \( \zeta_1, \zeta_2, \ldots \)) of the quantities

\[ \theta(\zeta, \zeta', x) \wedge \theta(\zeta'', \zeta''', x) \wedge \cdots , \]

where

\[ \theta(\zeta, \zeta', x) = R(\zeta, x)d_x R(\zeta', x) \]

and \( \{ \zeta, \zeta', \ldots \} \subset \{ \zeta_1, \zeta_2, \ldots \} \).
The 1-form \( \theta_j(\zeta, \zeta', x) = R(\zeta, x)dx R(\zeta', x) \) is our elementary building block for construction of invariants. We have

\[
\theta_j^{(k,k')} (\zeta, \zeta', x) = \sum_{k=-\infty}^{\infty} \sum_{k'=-\infty}^{\infty} \theta_j^{(k,k')} (x)(\zeta - \lambda_j(x))^k (\zeta' - \lambda_j(x))^{k'},
\]

\[
\theta_j^{(k,k')} (x) = \frac{1}{(2\pi i)^2} \int_{C_j(x)} \int_{C_j'(x)} \frac{R(\zeta, x)dx R(\zeta', x)}{(\zeta - \lambda_j(x))^{k+1}(\zeta' - \lambda_j(x))^{k'+1}} d\zeta' d\zeta,
\]

where the non-intersecting positively-oriented simple contours \( C_j(x) \) and \( C'_j(x) \) enclose the point \( \zeta = \lambda_j(x) \) and excluding all other eigenvalues of \( T(x) \). Without loss of generality, we choose the contour \( C'_j(x) \) to enclose the contour \( C_j(x) \).
Differential forms from the resolvent

Expand the resolvents in the Laurent series and use the Cauchy integral theorem to find

\[
\theta_j^{(k,k')}(x) = \frac{1}{(2\pi i)^2} \int_{C_j(x)} \frac{d\zeta'}{(\zeta' - \lambda_j(x))^{k'+1}} \sum_{l=-\infty}^{\infty} \int_{C_j(x)} \frac{R_j^{(l)}(x)d_x R(\zeta', x)d\zeta}{(\zeta - \lambda_j(x))^{k+1-l}}
\]

\[
= \frac{1}{2\pi i} \int_{C_j(x)} \frac{d\zeta'}{(\zeta' - \lambda_j(x))^{k'+1}} R_j^{(k)}(x)d_x R(\zeta', x)
\]

\[
= \frac{1}{2\pi i} \int_{C_j(x)} \frac{d\zeta'}{(\zeta' - \lambda_j(x))^{k'+1}} R_j^{(k)}(x) \sum_{l'=-\infty}^{\infty} \left(\frac{d_x R_j^{(l')}(x)(\zeta' - \lambda_j(x))^{l'}}{l'} \right)
\]

\[
- R_j^{(l')}(x) d_x \lambda_j(x) l' (\zeta' - \lambda_j(x))^{l'-1}
\]

\[
\theta_j^{(k,k')}(x) = R_j^{(k)}(x) \left(d_x R_j^{(k')}(x) - (k' + 1) R_j^{(k'+1)}(x) d_x \lambda_j(x) \right).
\]
Differential forms from the resolvent

The holomorphic function $\theta(\zeta, \zeta', x)$ is a generating function for 1-form invariants.
Examples:

$$\theta_{j}^{(-2,-2)}(x) = N_{j}(x)d_{x}N_{j}(x) + N_{j}(x)d_{x}\lambda_{j}(x),$$

$$\theta_{j}^{(-2,-1)}(x) = N_{j}(x)d_{x}P_{j}(x),$$

$$\theta_{j}^{(-2,0)}(x) = N_{j}(x)d_{x}A_{j}(x),$$

$$\theta_{j}^{(-1,-2)}(x) = P_{j}(x)d_{x}N_{j}(x) + P_{j}(x)d_{x}\lambda_{j}(x),$$

$$\theta_{j}^{(-1,-1)}(x) = P_{j}(x)d_{x}P_{j}(x),$$

$$\theta_{j}^{(-1,0)}(x) = P_{j}(x)d_{x}A_{j}(x),$$

$$\theta_{j}^{(0,-1)}(x) = A_{j}(x)d_{x}P_{j}(x),$$

The holomorphic function $\theta(\zeta, \zeta', x) \wedge \theta(\zeta'', \zeta''', x)$ is a generating function for 2-form invariants.
Example: \( n = 2, \, m = 1, \, m' = 1 \)

\[
T(x) = \begin{bmatrix}
x_4 + x_3 & x_1 - ix_2 \\
x_1 + ix_2 & x_4 - x_3
\end{bmatrix}, \; x_1, x_2, x_3, x_4 \in \mathbb{R}
\]

\[
\lambda_{\pm}(x) = x_4 \pm r, \quad r = (x_1^2 + x_2^2 + x_3^2)^{1/2}
\]

\( \lambda_+ \neq \lambda_-, \; r \neq 0 \)

\[
R(\zeta, x) = ((\zeta - x_4)^2 - r^2)^{-1} \begin{bmatrix}
\zeta - x_4 + x_3 & x_1 - ix_2 \\
x_1 + ix_2 & \zeta - x_4 - x_3
\end{bmatrix}
\]
Example: $n = 2, m = 1, m' = 1$

\[ R^{(k)}_{\pm}(x) = 0, \quad k \leq -2, \]

\[ R^{(-1)}_{\pm}(x) = \pm \frac{1}{2r} \begin{bmatrix} x_3 \pm r & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \pm r \end{bmatrix}, \]

\[ R^{(k)}_{\pm}(x) = -\frac{(\mp 1)^k}{(2r)^{k+2}} \begin{bmatrix} x_3 \mp r & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \mp r \end{bmatrix}, \quad k \geq 0. \]

\[ P_{\pm}(x) = \pm \frac{1}{2r} \begin{bmatrix} x_3 \pm r & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \pm r \end{bmatrix}, \quad N_{\pm}(x) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \]

\[ A_{\pm}(x) = -\frac{1}{(2r)^2} \begin{bmatrix} x_3 \mp r & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \mp r \end{bmatrix} \]
Example: $n = 2$, $m = 1$, $m' = 1$

$$
\Phi_\pm(x) = \frac{i}{4r^4} \left[ \begin{array}{cc} x_3 \pm r & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \pm r \end{array} \right] \sigma(x),
$$

$$
\sigma(x) = x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_3 dx_1 \wedge dx_2.
$$

$$
Q^{(1)}_{2,\pm} = \text{tr } \Phi_\pm(x) = \pm \frac{i}{2r^3} \sigma(x)
$$

$$
V_\pm(x) = \frac{e^{i\varphi_\pm(x)}}{(2r(r \pm x_3))^{1/2}} \left[ \begin{array}{c} x_3 \pm r \\ x_1 + ix_2 \end{array} \right], \text{ where } \varphi_\pm : D \rightarrow \mathbb{R} \text{ are arbitrary}
$$

$$
\omega_\pm(x) = \frac{i(x_1 dx_2 - x_2 dx_1)}{2r(r \pm x_3)} + i\varphi_\pm(x)
$$

$$
\Omega_\pm(x) = \pm \frac{i}{2r^3} \sigma(x)
$$

$P$, $N$, $\Phi$, $\omega$, $\Omega$, $Q^{(1)}_\pm$ are homogeneous functions of degree 0.
Example: $n = 2, m = 2, m' = 1$

$$T(x) = \begin{bmatrix} x_4 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_4 - x_3 \end{bmatrix}, \ x_1, x_2, x_3, x_4 \in \mathbb{C}$$

$$\lambda_{\pm}(x) = x_4 \pm r, \quad r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$$

$$\lambda_+ = \lambda_-, \ r = 0$$

$$R(\zeta, x) = (\zeta - x_4)^{-2} \begin{bmatrix} \zeta - x_4 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & \zeta - x_4 - x_3 \end{bmatrix}$$
Example: $n = 2, m = 2, m' = 1$

\[ R^{(k)}_\pm (x) = 0, \ k \leq -3, \]

\[ R^{(-2)}_\pm (x) = \begin{bmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{bmatrix}, \]

\[ R^{(-1)}_\pm (x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \]

\[ R^{(k)}_\pm (x) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \ k \geq 0. \]

\[ P_\pm (x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ N_\pm (x) = \begin{bmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{bmatrix}, \ A_\pm (x) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \]
Example: $n = 2, m = 2, m' = 1$

The only invariant 2-form is

$$N_{\pm}(x)dN_{\pm}(x) \wedge dN_{\pm}(x)N_{\pm}(x) = 2iN_{\pm}(x)\sigma(x)$$

$$= \begin{bmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{bmatrix} \sigma(x).$$

$$\text{tr} \left( N_{\pm}(x)dN_{\pm}(x) \wedge dN_{\pm}(x)N_{\pm}(x) \right) = 0$$

Once again, all invariant 2-form are zero.

Nothing more to be done for the case $n = 2$.

Preliminary results for $n = 3, m = 2$ are also negative: there are no nonzero invariant 2-form.

Of course, no problem for $m = 1$, as $Q_2^{(1)} = \text{tr} \Omega \neq 0$. 
Summary and next steps

For a given eigenvalue of a linear transformation, there are two associated operators — the eigenprojection and eigennilpotent. For each eigenvector, there are corresponding connection and curvature forms. Integrals of the resulting invariant differential forms can be viewed as generalizations of the geometric phase.

Find all invariants for $n = 3$ by both algebraic and analytic methods. Can all invariants be found directly from the resolvent? Are there additional advantages (besides the computational convenience) for the resolvent method? How to deal with degeneracies not in $D$? Can we define geometric invariants associated with more than one eigenvalue at the same time?