On non-geometric backgrounds and non-abelian T-duality

Erik Plauschinn
University of Padova

Miami 2014 — 12/22/2014
This talk is based on ::

- *T-duality revisited* [arXiv:1310.4194]
- *On T-duality transformations for the three-sphere* [arXiv:1408.1715]
Non-geometric backgrounds
Non-geometric backgrounds :: why?
Non-geometric backgrounds :: why?

- Can have non-commutative or non-associative features.
- Are part of the string-theory landscape.
- Provide uplifts for gauged supergravity theories.
- Can help with moduli stabilization & cosmology.

...
What is a non-geometric background?
What is a non-geometric background? ... apply T-duality to a three-torus:

\[
\begin{align*}
& H_{abc} \quad \leftrightarrow \quad T_c \\
& f_{ab}^c \quad \leftrightarrow \quad T_b \\
& Q_{a}{}^{bc} \quad \leftrightarrow \quad T_a \\
& R^{abc}
\end{align*}
\]

flux background \quad "twisted torus" \quad T-fold \quad non-associative
T-duality

- The physics of string theory **compactified** on two **circles**
  
  
  is indistinguishable.

- The **Buscher rules** transform between these backgrounds.
What is a non-geometric background? … apply T-duality to a three-torus:

\[ H_{abc} \leftrightarrow T_c \rightarrow f_{ab}^c \leftrightarrow T_b \rightarrow Q_a^{bc} \leftrightarrow T_a \rightarrow R^{abc} \]

- flux background
- "twisted torus"
- T-fold
- non-associative

Shelton, Taylor, Wecht - 2005
$H_{abc} \overset{T_c}{\longleftarrow} f_{ab}^c \overset{T_b}{\longleftarrow} Q_{a}^{bc} \overset{T_a}{\longleftarrow} R_{abc}$
Consider string theory compactified on a **three-torus** with *H*-flux:

- The geometry is determined by
  \[ ds^2 = dx^2 + dy^2 + dz^2 , \]
  \[ B_{yz} = N x , \]
  \[ x \sim x + 1 , \quad y \sim y + 1 , \quad z \sim z + 1 . \]

- The *H*-flux reads
  \[ H_{xyz} = N . \]
Consider string theory compactified on a three-torus with $H$-flux:

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\[ ds^2 = dx^2 + dy^2 + dz^2 , \]
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Consider string theory compactified on a three-torus with $H$-flux:

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- The $H$-flux reads
  
  \[ H_{xyz} = N . \]
After a T-duality in the $z$-direction, one arrives at a twisted torus:

- The geometry is determined by
  \[ ds^2 = dx^2 + dy^2 + (dz + Nxdy)^2 , \]
  \[ B = 0 , \]
  \[ (x, z) \sim (x + 1, z - Ny) , \quad y \sim y + 1 , \quad z \sim z + 1 . \]

- The geometric flux arises via
  \[ e^x = dx , \quad e^y = dy , \quad e^z = dz + Nxdy , \]
  \[ \omega^z_{xy} = N/2 , \]
  \[ [e_x, e_y] = -N e_z . \]
After a T-duality in the $z$-direction, one arrives at a twisted torus:

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After a second T-duality in the x-direction, one arrives at a **T-fold**:

- The geometry is determined by
  \[
  ds^2 = dy^2 + \frac{1}{1 + (Ny)^2} (dx^2 + dz^2),
  \]
  \[
  B_{xz} = \frac{Ny}{1 + (Ny)^2},
  \]
  \[
  y \sim y + 1, \quad z \sim z + 1.
  \]

- The non-geometric flux reads
  \[
  Q_{y}^{xz} = -N.
  \]

This space is **locally geometry**, but **globally non-geometric**.
After a second T-duality in the x-direction, one arrives at a T-fold:

- The geometry is determined by

\[
ds^2 = dy^2 + \frac{1}{1 + (N y)^2} (dx^2 + dz^2),
\]

\[
B_{xz} = \frac{N y}{1 + (N y)^2},
\]

\[
y \sim y + 1, \quad z \sim z + 1.
\]
After a second T-duality in the $x$-direction, one arrives at a **T-fold**:

- **The geometry is determined by**
  
  $$ds^2 = dy^2 + \frac{1}{1 + (N y)^2} (dx^2 + dz^2) ,$$
  
  $$B_{xz} = \frac{N y}{1 + (N y)^2} ,$$
  
  $$y \sim y + 1 , \quad z \sim z + 1 .$$

- **The non-geometric flux reads**
  
  $$Q_y^{xz} = -N .$$

This space is **locally geometry**, but **globally non-geometric**.
After formally applying a third T-duality, one obtains an \( R \)-flux background:

- The geometry is not even locally defined.

- The non-geometric \( R \)-flux is obtained by raising the index of the \( Q \)-flux:

  \[ Q_{y^{xz}} \rightarrow R^{xyz} = N. \]

- This background gives rise to a non-associative structure.
Summary :: Non-geometric backgrounds originate from

- applying T-duality transformations
- to ordinary backgrounds with $H$-flux.
Summary :: **Non-geometric** backgrounds originate from
- applying T-duality transformations
- to ordinary backgrounds with **H-flux**.

But :: ...
Summary :: Non-geometric backgrounds originate from
- applying T-duality transformations
- to ordinary backgrounds with H-flux.

But :: …
- The torus is the mostly-studied background.
- Other – and better – examples are needed!
→ Consider the three-sphere.
Aim ::
- Construct new non-geometric backgrounds.

Plan ::
- Revisit (collective) T-duality.
- Review the three-torus.
- Consider the three-sphere.
1. introduction
2. collective t-duality
3. three-torus
4. three-sphere
5. discussion
1. introduction

2. collective t-duality

3. three-torus

4. three-sphere

5. discussion
To study T-duality for the three-sphere, a non-abelian version might be needed.
To study T-duality for the **three-sphere**, a **non-abelian** version might be needed.
Consider the sigma-model action for the NS-NS sector of the closed string

\[ S = -\frac{1}{4\pi\alpha'} \int_{\partial\Sigma} \left[ G_{ij} \, dX^i \wedge \ast dX^j + \alpha' R \phi \ast 1 \right] - \frac{i}{2\pi\alpha'} \int_{\Sigma} \frac{1}{3!} H_{ijk} dX^i \wedge dX^j \wedge dX^k . \]

This action is invariant under global transformations \( \delta \epsilon X^i = \epsilon^{\alpha} k^i_{\alpha}(X) \) if

\[ \mathcal{L}_{k^i_{\alpha}} G = 0 \, , \quad \iota_{k^i_{\alpha}} H = d\nu^i_{\alpha} \, , \quad \mathcal{L}_{k^i_{\alpha}} \phi = 0 \, . \]

In general, the isometry algebra is non-abelian :: \([k^i_{\alpha}, k^i_{\beta}]_L = f_{\alpha\beta\gamma}^i k^i_{\gamma}\).
Consider the sigma-model action for the NS-NS sector of the closed string

\[
S = -\frac{1}{4\pi\alpha'} \int_{\partial\Sigma} \left[ G_{ij} \, dX^i \wedge \ast dX^j + \alpha' R \phi \ast 1 \right] - \frac{i}{2\pi\alpha'} \int_\Sigma \frac{1}{3!} H_{ijk} \, dX^i \wedge dX^j \wedge dX^k .
\]

This action is invariant under global transformations \( \delta \epsilon X^i = \epsilon^\alpha k^i_\alpha (X) \) if

\[
\mathcal{L}_{k_\alpha} G = 0 , \quad \iota_{k_\alpha} H = dv_\alpha , \quad \mathcal{L}_{k_\alpha} \phi = 0 .
\]

In general, the isometry algebra is non-abelian :: \([k_\alpha, k_\beta]_L = f_{\alpha\beta}{}^\gamma k_\gamma\).
Following Buscher’s procedure, one first gauges isometries of the action

\[
\hat{S} = - \frac{1}{2\pi \alpha'} \int_{\partial \Sigma} \frac{1}{2} G_{ij} (dX^i + k^i_\alpha A^\alpha) \wedge * (dX^j + k^j_\beta A^\beta) \\
- \frac{i}{2\pi \alpha'} \int_{\Sigma} \frac{1}{3!} H_{ijk} dX^i \wedge dX^j \wedge dX^k \\
- \frac{i}{2\pi \alpha'} \int_{\partial \Sigma} \left[ (v_\alpha + d\chi_\alpha) \wedge A^\alpha + \frac{1}{2} (\iota_{k^i_\alpha} v^i_\beta + f_{\alpha \beta}^\gamma \chi_\gamma) A^\alpha \wedge A^\beta \right].
\]

The local symmetry transformations and constraints take the form

\[
\hat{\delta}_\epsilon X^i = \epsilon^\alpha k^i_\alpha, \\
\hat{\delta}_\epsilon A^\alpha = -d\epsilon^\alpha - \epsilon^\beta A^\gamma f_{\beta \gamma}^\alpha, \\
\hat{\delta}_\epsilon \chi_\alpha = -\iota_{k_{(\alpha}} v_{\beta)}^i \epsilon^i - f_{\alpha \beta}^\gamma \epsilon^\beta \chi_\gamma,
\]

\[
\mathcal{L}_{k^i_\alpha} = f_{\alpha \beta}^\gamma v^\gamma, \\
3 \iota_{k^i_\alpha} f_{\beta \gamma}^i \delta v^\gamma = \iota_{k^i_\alpha} \iota_{k^j_\beta} \iota_{k^\gamma} H.
\]
The original model is recovered via the equations of motion for $\chi_\alpha$

$$0 = dA^\alpha - \frac{1}{2} f_{\beta\gamma}{}^\alpha A^\beta \wedge A^\gamma.$$

The gauged action can then be rewritten in terms of $DX^i = dX^i + k^i_\alpha A^\alpha$ as

$$\tilde{S} = - \frac{1}{4\pi\alpha'} \int_{\partial\Sigma} \left[ G_{ij} DX^i \wedge *DX^j + \alpha' R \phi \star 1 \right]$$

$$- \frac{i}{2\pi\alpha'} \int_{\Sigma} \frac{1}{3!} H_{ijk} DX^i \wedge DX^j \wedge DX^k.$$

Ignoring technical details, one replaces $DX^i \rightarrow dY^i$ and obtains the ungauged action.
The dual model is obtained via the equations of motion for $A^\alpha$

$$A^\alpha = -\left([G - \mathcal{D} G^{-1} \mathcal{D}]^{-1}\right)^{\alpha\beta} \left(1 + i \star \mathcal{D} G^{-1}\right)^\beta (k + i \star \xi)^\gamma,$$

where

$$G_{\alpha\beta} = k^i_\alpha G_{ij} k^j_\beta, \quad \xi_\alpha = d\chi_\alpha + \nu_\alpha,$$

$$\mathcal{D}_{\alpha\beta} = \iota_{k_\alpha} \nu_\beta + f_{\alpha\beta} \gamma \chi_\gamma, \quad k_\alpha = k^i_\alpha G_{ij} dX^j.$$

The action of the dual sigma-model is found by integrating-out $A^\alpha$ and reads

$$\tilde{S} = -\frac{1}{4\pi\alpha'} \int_{\partial \Sigma} \left[\tilde{G} + \alpha' R \phi \star 1\right] - \frac{i}{2\pi\alpha'} \int_{\Sigma} \tilde{H},$$

where, with $\mathcal{M} = G - \mathcal{D} G^{-1} \mathcal{D}$ invertible,

$$\tilde{G} = G + \begin{pmatrix} k \\ \xi \end{pmatrix}^T \begin{pmatrix} -\mathcal{M}^{-1} & -\mathcal{M}^{-1} \mathcal{D} G^{-1} \\ +\mathcal{M}^{-1} \mathcal{D} G^{-1} & +\mathcal{M}^{-1} \end{pmatrix} \wedge \star \begin{pmatrix} k \\ \xi \end{pmatrix},$$

$$\tilde{H} = H + \frac{1}{2} d \begin{pmatrix} k \\ \xi \end{pmatrix}^T \begin{pmatrix} +\mathcal{M}^{-1} \mathcal{D} G^{-1} & +\mathcal{M}^{-1} \\ -\mathcal{M}^{-1} & -\mathcal{M}^{-1} \mathcal{D} G^{-1} \end{pmatrix} \wedge \begin{pmatrix} k \\ \xi \end{pmatrix}.$$
Consider an enlarged target-space parametrized by coordinates $X^i$ and $\chi_\alpha$.

The enlarged metric $\tilde{G}$ and field strength $\tilde{H}$ have null-eigenvectors (and isometries)

$$\nu \tilde{n}_\alpha \tilde{G} = 0,$$
$$\nu \tilde{n}_\alpha \tilde{H} = 0,$$
$$\tilde{n}_\alpha = k_\alpha + D_{\alpha\beta} \partial \xi_\beta.$$

The dual metric and field strength are obtained via a change of coordinates

$$\mathcal{T}^I_A = \begin{pmatrix} k & 0 \\ D & 1 \end{pmatrix},$$
$$\tilde{G}_{AB} = (\mathcal{T}^T \tilde{G} \mathcal{T})_{AB} = \begin{pmatrix} 0 & 0 \\ 0 & G_{\alpha\beta} \end{pmatrix},$$
$$\tilde{H}_{ABC} = \tilde{H}_{IJK} \mathcal{T}^I_A \mathcal{T}^J_B \mathcal{T}^K_C,$$
$$\tilde{H}_{iBC} = 0.$$
The T-duality transformation rules are obtained via Buscher’s procedure of

1. **gauging** isometries in the sigma-model action,
2. **integrating-out** the gauge field,
3. performing a **change of coordinates**.

The possible gaugings are **restricted** by (recall that $\iota_{k\alpha} H = dv_\alpha$)

$$
\mathcal{L}_{k_{[\alpha} v_{\beta]}} = f_{\alpha\beta\gamma} v_\gamma, \quad
\iota_{k_{[\alpha}} f_{\beta\gamma]} \delta v_\delta = \frac{1}{3} \iota_{k\alpha} \iota_{k\beta} \iota_{k\gamma} H.
$$

The change of coordinates is performed using **null-eigenvectors** $\tilde{n}_\alpha$

$$
\tilde{G}_{IJ} \tilde{n}_\alpha^J = 0, \quad \tilde{H}_{IK} \tilde{n}_\alpha^K = 0.
$$
1. introduction
2. collective t-duality
3. three-torus
4. three-sphere
5. discussion
Consider a three-torus with $H$-flux specified as follows

$$ds^2 = R_1^2 (dX^1)^2 + R_2^2 (dX^2)^2 + R_3^2 (dX^3)^2,$$

$$H = h dX^1 \wedge dX^2 \wedge dX^3,$$

$$X^i \simeq X^i + \ell_s,$$

$$h \in \ell_s^{-1} \mathbb{Z}.$$

The Killing vectors are abelian and can be chosen as

$$k_1 = \partial_1,$$
$$k_2 = \partial_2,$$
$$k_3 = \partial_3.$$

The one-forms $\nu_\alpha$ are defined via $\iota_{k_\alpha} H = d\nu_\alpha$. 
Consider **one T-duality** along $k_1 = \partial_1$. The corresponding one-form reads

$$v = h\alpha X^2 dX^3 - h(1 - \alpha)X^3 dX^2, \quad \alpha \in \mathbb{R}.$$  

The **constraints** for gauging the sigma-model are trivially satisfied.

The geometry of the **dual background** is determined from the quantities

$$G = R_1^2, \quad \xi = d\chi + v, \quad D = 0, \quad k = R_1^2 dX^1, \quad \mathcal{M} = G = R_1^2.$$

The metric and field strength are obtained as …
Consider one T-duality along $k_1 = \partial_1$. The corresponding one-form reads

$$v = h \alpha X^2 dX^3 - h (1 - \alpha) X^3 dX^2, \quad \alpha \in \mathbb{R}.$$ 

The constraints for gauging the sigma-model are trivially satisfied.

The geometry of the dual background is determined from the quantities

$$\mathcal{G} = R^2_1, \quad \xi = d\chi + v, \quad \mathcal{D} = 0, \quad k = R^2_1 dX^1, \quad \mathcal{M} = \mathcal{G} = R^2_1.$$ 

The metric and field strength are obtained as …
Consider one T-duality along $k_1 = \partial_1$. The corresponding one-form reads

$$v = h^{\imath}X^2 dX^3 h (1^{\imath})X^3 dX^2,$$

The constraints for gauging the sigma-model are trivially satisfied.

The geometry of the dual background is determined from the quantities

$$M = G = R^2, G = R^2, \imath = d + v, D = 0, k = R^2 dX^1,$$

The metric and field strength are obtained as ...
Consider one T-duality along \( k_1 = \partial_1 \). The corresponding one-form reads

\[
\kappa = \partial_1, \quad \nu = h \iota_X^2 dX^3 = h (1 + \iota_X) X^3 dX^2, \quad 2 \mathcal{R}.
\]

The constraints for gauging the sigma-model are trivially satisfied.

The geometry of the dual background is determined from the quantities

\[
M = G = R^2, \quad \iota_X = d + \nu, \quad D = 0, \quad \kappa = R^2 dX^1.
\]

The metric and field strength are obtained as:

\[
\tilde{\mathcal{G}} = G + \begin{pmatrix} k \\ \xi \end{pmatrix}^T \begin{pmatrix} -M^{-1} & -M^{-1}DG^{-1} \\ +M^{-1}DG^{-1} & +M^{-1} \end{pmatrix} \wedge \star \begin{pmatrix} k \\ \xi \end{pmatrix}
\]

\[
\xi = d\chi + \nu.
\]
Consider one T-duality along $k_1 = \partial_1$. The corresponding one-form reads

$$
\xi = d\chi + v
$$

The geometry of the dual background is determined from the quantities

$$
M = G = R^2_1, \quad G = R^2_1, \quad \xi = d + v,
$$

The metric and field strength are obtained as ...
Consider one T-duality along $k_1 = \partial_1$. The corresponding one-form reads

\[ k_1 = \partial_1, \quad v = h \bar{\xi} X^2 dX^3 - h(1 - \bar{\xi}) X^3 dX^2, \]

The constraints for gauging the sigma-model are trivially satisfied.

The geometry of the dual background is determined from the quantities $G = R_2^{-1}$, $\bar{\xi} = d\chi + v$.

The metric and field strength are obtained as

\[
\tilde{G} = G + \left( \begin{array}{c} k \\ \xi \end{array} \right)^T \left( -\mathcal{M}^{-1} + \mathcal{M}^{-1} \mathcal{D} \mathcal{G}^{-1} \right) \wedge \star \left( \begin{array}{c} k \\ \xi \end{array} \right)
\]

\[ = G + \left( \begin{array}{c} R_1^2 dX^1 \\ \xi \end{array} \right)^T \left( -\frac{1}{R_1^2} 0 \\ 0 + \frac{1}{R_1^2} \right) \wedge \star \left( \begin{array}{c} R_1^2 dX^1 \\ \xi \end{array} \right)
\]

\[ = G - R_1^2 dX^1 \wedge \star dX^1 + \frac{1}{R_1^2} \xi \wedge \star \xi \]

$\xi = d\chi + v$
Consider one T-duality along $k_1 = \partial_1$. The corresponding one-form reads:

\[ \xi = d\chi + v \]

The constraints for gauging the sigma-model are trivially satisfied. The geometry of the dual background is determined from the quantities:

\[ M = G = R_2 \]

\[ G = R_2, \quad \xi = d + v, \quad D = 0, \quad k = R_2 \partial_1 \]

The metric and field strength are obtained as:

\[
\mathcal{G} = G + \begin{pmatrix} k \\ \xi \end{pmatrix}^T \begin{pmatrix} -M^{-1} & -M^{-1}DG^{-1} \\ +M^{-1}DG^{-1} & +M^{-1} \end{pmatrix} \wedge \star \begin{pmatrix} k \\ \xi \end{pmatrix} \\
= G + \begin{pmatrix} R_2^2dX^1 \\ \xi \end{pmatrix}^T \begin{pmatrix} -\frac{1}{R_1^2} & 0 \\ 0 & +\frac{1}{R_1^2} \end{pmatrix} \wedge \star \begin{pmatrix} R_1^2dX^1 \\ \xi \end{pmatrix} \\
= G - R_1^2dX^1 \wedge \star dX^1 + \frac{1}{R_1^2} \xi \wedge \star \xi \\
= \frac{1}{R_1^2} \xi \wedge \star \xi + R_2^2dX^2 \wedge \star dX^2 + R_3^2dX^3 \wedge \star dX^3
\]
Consider one T-duality along $k_1 = \partial_1$. The corresponding one-form reads $v = h \wedge X^2 \wedge dX^3$. The constraints for gauging the sigma-model are trivially satisfied. The geometry of the dual background is determined from the quantities $M = G = R^2_1$. $G = R^2_1, \varepsilon = d + v, D = 0, k = R^2_1 \wedge dX^1$. The metric and field strength are obtained as ...
Consider one T-duality along $k_1 = \partial_1$. The corresponding one-form reads
\[ d v = \omega^2 = dX^3 \wedge \omega^1, \]

The constraints for gauging the sigma-model are trivially satisfied.

The geometry of the dual background is determined from the quantities
\[ M_1 = G = \Omega_2 - 1, \]
\[ \Omega = d + v, \]
\[ D = 0, \]
\[ k = \Omega_1 \wedge (k), \]

The metric and field strength are obtained as…

\[
\hat{H} = H + \frac{1}{2} d \left[ \begin{pmatrix} k \\ \xi \end{pmatrix}^T \begin{pmatrix} +M^{-1}DG^{-1} & +M^{-1} \\ -M^{-1} & -M^{-1}DG^{-1} \end{pmatrix} \wedge \begin{pmatrix} k \\ \xi \end{pmatrix} \right]
\]
Consider one T-duality along $k_1 = \partial_1$. The corresponding one-form reads:

$$\tilde{v} = h(\xi \cdot X^2 dX^3)$$

The constraints for gauging the sigma-model are trivially satisfied.

The geometry of the dual background is determined from the quantities $M = G = R^2$. $G = R^2, \xi = d + v, D = 0, k = R^2 dX^1$.

The metric and field strength are obtained as:

$$\tilde{H} = H + \frac{1}{2} d \left[ \left( \begin{array}{c} k \\ \xi \end{array} \right)^T \left( \begin{array}{cc} +M^{-1}DG^{-1} & +M^{-1} \\ -M^{-1} & -M^{-1}DG^{-1} \end{array} \right) \right] \wedge \left( \begin{array}{c} k \\ \xi \end{array} \right)$$

$$= H + \frac{1}{2} d \left[ \left( \begin{array}{c} R^2 dX^1 \\ \xi \end{array} \right)^T \left( \begin{array}{cc} 0 & +\frac{1}{R^2_1} \\ -\frac{1}{R^2_1} & 0 \end{array} \right) \right] \wedge \left( \begin{array}{c} R^2 dX^1 \\ \xi \end{array} \right)$$
Consider one T-duality along $k_1 = \partial_1$. The corresponding one-form reads

$$k_1 = \partial_1$$

and

$$\nu = h \cdots = G = R^{-2}.$$ 

The constraints for gauging the sigma-model are trivially satisfied.

The geometry of the dual background is determined from the quantities

$$M = G = R^{-2},$$

$$G = R^{-2}, \varepsilon = d + v, \quad D = 0, \quad k = R^{-2} dX_1,$$

The metric and field strength are obtained as ...

\[
\begin{align*}
\tilde{H} &= H + \frac{1}{2} d \left[ \begin{pmatrix}
k \\
\xi
\end{pmatrix}^T \begin{pmatrix}
M^{-1} \mathcal{D} \mathcal{G}^{-1} & +M^{-1} \\
-M^{-1} & -M^{-1} \mathcal{D} \mathcal{G}^{-1}
\end{pmatrix} \wedge \begin{pmatrix}
k \\
\xi
\end{pmatrix} \right] \\
&= H + \frac{1}{2} d \left[ \begin{pmatrix}
R_1^2 dX^1 \\
\xi
\end{pmatrix}^T \begin{pmatrix}
0 & +\frac{1}{R_1^2} \\
-\frac{1}{R_1^2} & 0
\end{pmatrix} \wedge \begin{pmatrix}
R_1^2 dX^1 \\
\xi
\end{pmatrix} \right] \\
&= H + d \left[ dX^1 \wedge \xi \right]
\end{align*}
\]
Consider one T-duality along \( k_1 = \partial_1 \). The corresponding one-form reads

\[
\begin{align*}
\kappa^1 &= \partial_1 \\
\nu &= h^{\tau X} X^2 dX^3 \\
\tau &= d + v, \\
D &= 0, \\
\kappa &= R_1^2 dX^1,
\end{align*}
\]

The constraints for gauging the sigma-model are trivially satisfied.

The geometry of the dual background is determined from the quantities

\[
M = G = R_1^2, \\
G = R_1^2, \\
\tau = d + v, \\
D = 0, \\
\kappa = R_1^2 dX^1.
\]

The metric and field strength are obtained as:

\[
\tilde{H} = H + \frac{1}{2} d \left[ \left( \begin{array}{c} k \\ \xi \end{array} \right)^T \left( \begin{array}{cc} +M^{-1}DG^{-1} & +M^{-1} \\ -M^{-1} & -M^{-1}DG^{-1} \end{array} \right) \wedge \left( \begin{array}{c} k \\ \xi \end{array} \right) \right]
\]

\[
= H + \frac{1}{2} d \left[ \left( \begin{array}{c} R_1^2 dX^1 \\ \xi \end{array} \right)^T \left( \begin{array}{cc} 0 & +\frac{1}{R_1^2} \\ -\frac{1}{R_1^2} & 0 \end{array} \right) \wedge \left( \begin{array}{c} R_1^2 dX^1 \\ \xi \end{array} \right) \right]
\]

\[
= H + d \left[ dX^1 \wedge \xi \right]
\]

\[
= 0
\]

\[
d\xi = d(d\chi + v) = h dX^2 \wedge dX^3
\]
Consider one T-duality along $k_1 = \partial_1$. The corresponding one-form reads

$$v = h \alpha X^2 dX^3 - h (1 - \alpha) X^3 dX^2, \quad \alpha \in \mathbb{R}.$$ 

The constraints for gauging the sigma-model are trivially satisfied.

The geometry of the dual background is determined from the quantities

$$G = R_1^2, \quad \xi = d\chi + v, \quad D = 0, \quad k = R_1^2 dX^1, \quad \rightarrow \quad M = G = R_1^2.$$ 

The metric and field strength are obtained as …
Consider one T-duality along $k_1 = \partial_1$. The corresponding one-form reads

$$v = h\alpha X^2 dX^3 - h(1 - \alpha)X^3 dX^2, \quad \alpha \in \mathbb{R}.$$ 

The constraints for gauging the sigma-model are trivially satisfied.

The geometry of the dual background is determined from the quantities

$$\mathcal{G} = R_1^2, \quad \xi = d\chi + v,$$

$$D = 0, \quad k = R_1^2 dX^1,$$

$$\mathcal{M} = \mathcal{G} = R_1^2.$$ 

The metric and field strength are obtained as

$$\tilde{\mathcal{G}} = \frac{1}{R_1^2} \xi \wedge \star \xi + R_2^2 dX^2 \wedge \star dX^2 + R_3^2 dX^3 \wedge \star dX^3,$$

$$\tilde{\mathcal{H}} = 0.$$
As expected, the dual background is a {**twisted torus**} (with $\alpha = 1$)

$$\tilde{d}s^2 = \frac{1}{R_1^2} \left( d\chi + h X^2 dX^3 \right)^2 + R_2^2 (dX^2)^2 + R_3^2 (dX^3)^2,$$

$$\tilde{H} = 0.$$
Consider two collective T-dualities along \( k_1 = \partial_1 \) and \( k_2 = \partial_2 \).

The constraints on gauging the sigma-model imply (for \( \alpha \in \mathbb{R} \))

\[
\begin{align*}
v_1 &= h\alpha X^2 dX^3 - h(1 - \alpha) X^3 dX^2, \\
v_2 &= h(1 + \alpha) X^3 dX^1 + h\alpha X^1 dX^3.
\end{align*}
\]

The geometry of the dual background is determined from

\[
\begin{align*}
\mathcal{G}_{\alpha\beta} &= \begin{pmatrix} R_1^2 & 0 \\ 0 & R_2^2 \end{pmatrix}, & \xi_\alpha &= \begin{pmatrix} d\chi_1 + v_1 \\ d\chi_2 + v_2 \end{pmatrix}, \\
\mathcal{D}_{\alpha\beta} &= \begin{pmatrix} 0 & +hX^3 \\ -hX^3 & 0 \end{pmatrix}, & k_\alpha &= \begin{pmatrix} R_1^2 dX^1 \\ R_2^2 dX^2 \end{pmatrix}.
\end{align*}
\]
The **metric** of the enlarged target space (in the basis \( \{dX^1, dX^2, dX^3, \xi_1, \xi_2 \} \)) reads

\[
\tilde{G}_{IJ} = \frac{1}{\rho} \begin{pmatrix}
\left[ R_1 h X^3 \right]^2 & 0 & 0 & 0 & -R_1^2 h X^3 \\
0 & \left[ R_2 h X^3 \right]^2 & 0 & +R_2^2 h X^3 & 0 \\
0 & 0 & \rho R_3^2 & 0 & 0 \\
0 & +R_2^2 h X^3 & 0 & R_2^2 & 0 \\
-R_1^2 h X^3 & 0 & 0 & 0 & R_1^2 \\
\end{pmatrix},
\]

\[
\rho = R_1^2 R_2^2 + \left[ h X^3 \right]^2.
\]

Performing then a **change of basis** one finds

\[
\mathcal{T}^I_A = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -h X^3 & 0 \\
0 & -h X^3 & 0 & 1 & 0 \\
\end{pmatrix}
\quad \rightarrow \quad
\tilde{g}_{AB} = (\mathcal{T}^T \tilde{G} \mathcal{T})_{AB} = \frac{1}{\rho} \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & \rho R_3^2 & 0 & 0 \\
0 & 0 & 0 & R_2^2 & 0 \\
0 & 0 & 0 & 0 & R_1^2 \\
\end{pmatrix}.
\]
Performing a similar analysis for the field strength and adjusting the notation, one finds

\[ \tilde{ds}^2 = \frac{1}{R_1^2 R_2^2 + [h X^3]^2} \left[ R_1^2 (d\tilde{\chi}_1)^2 + R_2^2 (d\tilde{\chi}_2)^2 \right] + R_3^2 (dX^3)^2, \]

\[ \tilde{H} = -\hbar \frac{R_1^2 R_2^2 - [h X^3]^2}{\left[R_1^2 R_2^2 + [h X^3]^2\right]^2} d\tilde{\chi}_1 \wedge d\tilde{\chi}_2 \wedge dX^3. \]

This is the familiar T-fold background.
Finally, consider three collective T-dualities along $k_1 = \partial_1$, $k_2 = \partial_2$ and $k_3 = \partial_3$.

The constraints on gauging the sigma-model require the $H$-flux to be vanishing

$$\tau_{k_\alpha} \tau_{k_\beta} \tau_{k_\gamma} H = 0 \quad \rightarrow \quad H = 0 .$$

The dual model is characterized by

$$\tilde{d}s^2 = \frac{1}{R_1^2} (d\chi_1)^2 + \frac{1}{R_2^2} (d\chi_2)^2 + \frac{1}{R_3^2} (d\chi_3)^2 ,$$

$$\tilde{H} = 0 .$$
Finally, consider three collective T-dualities along $k_1 = \partial_1$, $k_2 = \partial_2$ and $k_3 = \partial_3$.

The constraints on gauging the sigma-model require the $H$-flux to be vanishing

$$\kappa_{k_\alpha} \kappa_{k_\beta} \kappa_{k_\gamma} H = 0 \quad \Rightarrow \quad H = 0.$$
Finally, consider three collective T-dualities along $k_1 = \partial_1$, $k_2 = \partial_2$ and $k_3 = \partial_3$.

The constraints on gauging the sigma-model require the $H$-flux to be vanishing

$$
\iota_{k_\alpha} \iota_{k_\beta} \iota_{k_\gamma} H = 0 \quad \mapsto \quad H = 0.
$$

The dual model is characterized by

$$
\tilde{ds}^2 = \frac{1}{R_1^2} (d\chi_1)^2 + \frac{1}{R_2^2} (d\chi_2)^2 + \frac{1}{R_3^2} (d\chi_3)^2,
$$

$$
\tilde{H} = 0.
$$
The **formalism** for T-duality introduced above works as expected.

- **three-torus with $H$-flux**
  - 1 T-duality → twisted torus
  - 2 T-dualities → T-fold
  - 3 T-dualities → torus with $R \rightarrow 1/R$
1. introduction
2. collective t-duality
3. three-torus
4. three-sphere
5. discussion
Consider a three-sphere with $H$-flux, specified by

$$ds^2 = R^2 \left[ \sin^2 \eta (d\zeta_1)^2 + \cos^2 \eta (d\zeta_2)^2 + (d\eta)^2 \right], \quad \zeta_{1,2} = 0 \ldots 2\pi,$$

$$H = \frac{h}{2\pi^2} \sin \eta \cos \eta d\zeta_1 \wedge d\zeta_2 \wedge d\eta, \quad \eta = 0 \ldots \frac{\pi}{2}.$$

This model is conformal if $h = 4\pi^2 R^2$.

The isometry algebra is $\mathfrak{so}(4) = \mathfrak{su}(2) \times \mathfrak{su}(2)$, and the Killing vectors satisfy

(with $\alpha, \beta, \gamma \in \{1, 2, 3\}$)

$$[K_\alpha, K_\beta]_L = \epsilon_{\alpha \beta}^\gamma K_\gamma,$$

$$[K_\alpha, \tilde{K}_\beta]_L = 0, \quad |K_\alpha|^2 = |\tilde{K}_\alpha|^2 = \frac{R^2}{4}.$$

$$[\tilde{K}_\alpha, \tilde{K}_\beta]_L = \epsilon_{\alpha \beta}^\gamma \tilde{K}_\gamma.$$
Consider a three-sphere with $H$-flux, specified by

$$\alpha = 0 \ldots \frac{\pi}{2},$$

$$\beta , \gamma = 0 \ldots 2\pi,$$

$$H = n^2 \frac{\pi}{2} \sin \alpha \cos \alpha d \beta^1 \ldots d \beta^2 \alpha,$$

This model is conformal if

$$h = 4 \frac{\pi}{2} R^2.$$  

The isometry algebra is

$$[K_\alpha, K_\beta]_L = \epsilon_{\alpha \beta \gamma} K_\gamma,$$

$$[K_\alpha, \tilde{K}_\beta]_L = 0,$$

$$[\tilde{K}_\alpha, K_\beta]_L = \epsilon_{\alpha \beta \gamma} \tilde{K}_\gamma,$$

$$|K_\alpha|^2 = |\tilde{K}_\alpha|^2 = \frac{R^2}{4}.$$
Consider a **three-sphere** with **$H$-flux**, specified by

\[ ds^2 = R^2 \left[ \sin^2 \eta (d\zeta_1)^2 + \cos^2 \eta (d\zeta_2)^2 + (d\eta)^2 \right], \quad \zeta_{1,2} = 0 \ldots 2\pi, \]

\[ H = \frac{h}{2\pi^2} \sin \eta \cos \eta d\zeta_1 \wedge d\zeta_2 \wedge d\eta, \quad \eta = 0 \ldots \frac{\pi}{2}. \]

This model is **conformal** if \( h = 4\pi^2 R^2 \).

The isometry algebra is \( \mathfrak{so}(4) = \mathfrak{su}(2) \times \mathfrak{su}(2) \), and the **Killing vectors** satisfy 
(with \( \alpha, \beta, \gamma \in \{1, 2, 3\} \))

\[ [K_\alpha, K_\beta]_L = \epsilon_{\alpha\beta\gamma} K_\gamma, \]

\[ [K_\alpha, \tilde{K}_\beta]_L = 0, \quad |K_\alpha|^2 = |\tilde{K}_\alpha|^2 = \frac{R^2}{4}. \]

\[ [\tilde{K}_\alpha, \tilde{K}_\beta]_L = \epsilon_{\alpha\beta\gamma} \tilde{K}_\gamma, \]
Consider **one T-duality** along $K_1$. In this case, all **constraints** are **satisfied**:

- constraints from gauging the sigma-model
- the matrix $G_{\alpha\beta} = k_{\alpha}^i G_{ij} k_{\beta}^j$ is invertible

The **dual model** is characterized by the metric and $H$-flux:

$$\tilde{G} = \frac{R^2}{4} \left[ (d\tilde{\eta})^2 + \sin^2(\tilde{\eta}) (d\tilde{\zeta})^2 \right] + \frac{4}{R^2} \xi \wedge \ast \xi,$$

$$d\xi = -\frac{h}{16\pi^2} \sin \tilde{\eta} d\tilde{\eta} \wedge d\tilde{\zeta}.$$

This metric describes a **circle fibered over a two-sphere**.
For two collective T-dualities, consider the commuting Killing vectors $K_1$ and $\tilde{K}_1$.

The constraints for this model are almost satisfied:

- constraints from gauging the sigma-model $\checkmark$
- the matrix $G_{\alpha \beta} = k_\alpha^i G_{ij} k_\beta^j$ is invertible $\times$

The dual model, via the above formalism, takes a form similar to the T-fold

\[
\tilde{G} = R^2 (d\eta)^2 + \frac{1}{R^2} \frac{(d\tilde{\chi}_1)^2}{\sin^2 \eta + \left(\frac{h}{4\pi^2 R^2}\right)^2 \cos^2 \eta} + \frac{1}{R^2} \frac{(d\tilde{\chi}_2)^2}{\cos^2 \eta + \left(\frac{h}{4\pi^2 R^2}\right)^2 \cos^4 \eta},
\]

\[
\tilde{H} = -8 h \pi^2 (h^2 - 16 \pi^4 R^4) \frac{\sin \eta \cos \eta}{\left[16 \pi^2 R^4 \sin^2 \eta + h^2 \cos^2 \eta\right]^2} d\eta \wedge d\tilde{\chi}_1 \wedge d\tilde{\chi}_2.
\]
For two collective T-dualities, consider the commuting Killing vectors $K_1$ and $\tilde{K}_1$.

The constraints for this model are almost satisfied:

- constraints from gauging the sigma-model  
  ✓
- the matrix $G_{\alpha\beta} = k^{i}_\alpha G_{ij} k^{j}_\beta$ is invertible  
  ✗  $\det G = \frac{R^4}{16} \sin^2(2\eta)$

The dual model, via the above formalism, takes a form similar to the T-fold

\[
\tilde{G} = R^2 (d\eta)^2 + \frac{1}{R^2} \frac{(d\tilde{\chi}_1)^2}{\sin^2 \eta + \left[\frac{h}{4\pi^2 R^2}\right]^2 \cos^2 \eta} + \frac{1}{R^2} \frac{(d\tilde{\chi}_2)^2}{\cos^2 \eta + \left[\frac{h}{4\pi^2 R^2}\right]^2 \frac{\cos^4 \eta}{\sin^2 \eta}},
\]

\[
\tilde{H} = -8 h \pi^2 (h^2 - 16\pi^4 R^4) \frac{\sin \eta \cos \eta}{[16\pi^2 R^4 \sin^2 \eta + h^2 \cos^2 \eta]^2} d\eta \wedge d\tilde{\chi}_1 \wedge d\tilde{\chi}_2.
\]
For two collective T-dualities, consider the commuting Killing vectors $K_1$ and $\tilde{K}_1$.

The constraints for this model are almost satisfied:

- constraints from gauging the sigma-model $\checkmark$
- the matrix $G_{\alpha\beta} = k_{\alpha}^i G_{ij} k_{\beta}^j$ is invertible $\times$

The dual model, via the above formalism, takes a form similar to the T-fold

$$\tilde{G} = R^2 (d\eta)^2 + \frac{1}{R^2} \frac{(d\tilde{\chi}_1)^2}{\sin^2 \eta + \left[\frac{h}{4\pi^2 R^2}\right]^2 \cos^2 \eta} + \frac{1}{R^2} \frac{(d\tilde{\chi}_2)^2}{\cos^2 \eta + \left(\frac{h}{4\pi^2 R^2}\right)^2 \frac{\cos^4 \eta}{\sin^2 \eta}},$$

$$\tilde{H} = -8 h \pi^2 (h^2 - 16 \pi^4 R^4) \frac{\sin \eta \cos \eta}{\left[16 \pi^2 R^4 \sin^2 \eta + h^2 \cos^2 \eta\right]^2} d\eta \wedge d\tilde{\chi}_1 \wedge d\tilde{\chi}_2.$$

But, when starting from a conformal model with \( h = 4\pi^2 R^2 \), the background becomes

\[
\overline{\mathcal{G}} = R^2 (d\eta)^2 + \frac{1}{R^2} \left[ (d\tilde{\chi}_1)^2 + \tan^2 \eta (d\tilde{\chi}_2)^2 \right],
\]

\( \overline{H} = 0 \).

With dual dilaton \( \overline{\phi} = -\log (R^2 \cos \eta) + \phi \), this is again a conformal model.
Consider finally a non-abelian T-duality along $K_1$, $K_2$ and $K_3$.

- The constraints from gauging the sigma-model suggest $H=0$,
- and the matrix $G_{\alpha\beta} = k^i_{\alpha} k^j_{\beta} G_{ij}$ is invertible

The dual model is obtained as (with $\rho \geq 0$ and $\phi_{1,2} = 0, \ldots, 2\pi$)

\[
\hat{G} = \frac{4}{R^2} d\rho \wedge *d\rho + \frac{R^2}{4} \frac{\rho^2}{\rho^2 + \frac{R^4}{16}} \left[ d\phi_1 \wedge *d\phi_1 + \sin^2(\phi_1) d\phi_2 \wedge *d\phi_2 \right],
\]

\[
\hat{H} = \frac{\rho^2}{(\rho^2 + \frac{R^4}{16})^2} \left[ \rho^2 + 3 \frac{R^4}{16} \right] \sin(\phi_1) d\rho \wedge d\phi_1 \wedge d\phi_2.
\]
In the formalism for T-duality introduced above, for a conformal model one finds:

- A three-sphere with $H$-flux.
  - 1 T-duality: $S^1$ fibered over $S^2$.
  - 2 T-dualities: non-compact, geometric.
- A three-sphere with $H=0$.
  - 3 T-dualities: $S^2$ fibered over a ray.
1. introduction
2. collective t-duality
3. three-torus
4. three-sphere
5. discussion
Non-geometric features are mainly studied for the example of the torus.

- Other – and better – examples are needed!

  ➞ Consider the three-sphere.

An approach to collective T-duality has been developed,

- which does not require gauge-fixing, and

- which highlights new constraints.

Collective T-duality has been studied for two examples ::

- three-torus ➞ known results reproduced

- three-sphere ➞ no non-geometric backgrounds
For two collective T-duality transformations it was found ::

- three-torus with $H$-flux (not conformal)
  - 2 T-dualities
  - non-geometric (compact)

- three-sphere with $H$-flux (conformal)
  - 2 T-dualities
  - geometric (non-compact)
For two collective T-duality transformations it was found ::

- three-torus with $H$-flux (not conformal)
- three-sphere with $H$-flux (conformal)

2 T-dualities

- non-geometric compact
- geometric non-compact

Having a conformal model appears to be important for T-duality …

EP - work in progress