TOPOLOGICALLY MASSIVE YANG-MILLS THEORY

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Outline

- Chern-Simons Theory
- Topologically Massive Yang-Mills Theory
- The Classical Equivalence
- Skein Relations
The field equations for this action are

\[ \delta S_{CS} = \frac{k}{4\pi} \int_{\Sigma \times \{t_i, t_f\}} d^3 x \, \epsilon^{\mu \nu \rho} \, Tr[A_\mu \partial_\nu A_\alpha + \frac{2}{3} A_\mu A_\nu A_\alpha] \]

\[ A_\mu = -i A^a_\mu T^a \]

\[ [T^a, T^b] = f^{abc} T^c \]

for SU(N)

\[ a, b, c = 1, \ldots, N^2 - 1 \]

The field equations for this action are

\[ \frac{\delta S_{CS}}{\delta A^a_\mu} = \frac{k}{8\pi} \epsilon^{\mu \nu \rho} F^a_{\nu \rho} \Longrightarrow \epsilon^{\mu \nu \rho} F^a_{\nu \rho} = 0 \]

Where F is,

\[ F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \]
Chern-Simons Action

\[ S_{CS} = -\frac{k}{4\pi} \int_{\Sigma \times [t_i, t_f]} d^3x \, \epsilon^{\mu\nu\alpha} \text{Tr}[A_\mu \partial_\nu A_\alpha + \frac{2}{3} A_\mu A_\nu A_\alpha] \]

We choose temporal gauge and \( z = x - iy \), \( \bar{z} = x + iy \) coordinates.

\[ A_0 = 0 \quad A_z = \frac{1}{2} (A_1 + iA_2) \quad A_{\bar{z}} = \frac{1}{2} (A_1 - iA_2) \]

From field equations:

\[ F_{\mu\nu} = 0 \implies \begin{cases} 
F_{oz} = \partial_o A_z = 0 \\
F_{o\bar{z}} = \partial_o A_{\bar{z}} = 0 \\
F_{z\bar{z}} = \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z + [A_z, A_{\bar{z}}] = 0
\end{cases} \]
In the gauge we chose, the action becomes

\[ S_{\text{CS}} = -\frac{ik}{2\pi} \int dtd\mu_\Sigma \text{Tr}(A_\bar{z}\partial_0 A_z - A_z\partial_0 A_\bar{z}) \]

and the momenta are

\[ \Pi^z = \frac{ik}{2\pi} A_\bar{z} \quad \text{and} \quad \Pi^\bar{z} = -\frac{ik}{2\pi} A_z \]

Thus,

\[ \{A_z^a(z), A_w^b(w)\} = -\frac{2\pi i}{k} \delta^{ab} \delta^{(2)}(z - w) \]
The Wave functional for CS$^{[1]}$

Now, we choose a Bargmann polarization condition that makes

$$\Psi[A^a_z, A^a_{\bar{z}}] \rightarrow \Psi[A^a_{\bar{z}}]$$

which must satisfy the constraint

$$F^a_{z\bar{z}} \Psi[A^a_{\bar{z}}] = 0$$

$$\implies A^a_z \Psi[A^a_{\bar{z}}] = \frac{2\pi}{k'} \frac{\delta}{\delta A^a_{\bar{z}}} \Psi[A^a_{\bar{z}}]$$

For gauge group $SU(N)$ and $\Sigma = S^2$,

$$A_{\bar{z}} = -\partial_{\bar{z}}UU^{-1} \quad A_z = (U^{\dagger-1})\partial_z U^{\dagger}$$

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The Wave functional for CS

An infinitesimal gauge transformation on the wave functional

\[ \delta_{\epsilon} \Psi[A_{\bar{z}}] = \int d^2 z \, \delta_{\epsilon} A_{\bar{z}}^a \frac{\delta \Psi}{\delta A_{\bar{z}}^a} \]

\[ = \int d^2 z \epsilon^a \left( \partial_{\bar{z}} \frac{\delta}{\delta A_{\bar{z}}^a} + f^{abc} A_{\bar{z}}^b \frac{\delta}{\delta A_{\bar{z}}^c} \right) \Psi \]

then using

\[ A_{\bar{z}}^a \Psi[A_{\bar{z}}^a] = \frac{2\pi}{k} \frac{\delta}{\delta A_{\bar{z}}^a} \Psi[A_{\bar{z}}^a] \]

we get

\[ \delta_{\epsilon} \Psi[A_{\bar{z}}] = \frac{k}{2\pi} \int d^2 z \epsilon^a (F_{\bar{z} \bar{z}}^a - \partial_{\bar{z}} A_{\bar{z}}^a) \Psi \]

\[ = \frac{k}{2\pi} \int d^2 z \epsilon^a (\partial_{\bar{z}} A_{\bar{z}}^a) \Psi \]
The Wave functional for CS

\[ \delta_{\epsilon} \Psi = \frac{k}{2\pi} \int d^2 z \epsilon^a (\partial_z A^a_{\bar{z}}) \Psi \]

This is a well known condition and it is solved by

\[ \Psi[A_{\bar{z}}] = \exp(k S_{WZW}(U)) \quad | \quad A_{\bar{z}} = -\partial_{\bar{z}} U U^{-1} \]

where \( S_{WZW} \) is the Wess-Zumino-Witten action given by

\[ S_{WZW}(U) = \frac{1}{2\pi} \int_{\Sigma} d^2 z \, \text{tr} \, \partial_z U \partial_{\bar{z}} U^{-1} + \]

\[ - \frac{i}{12\pi} \int_V d^3 x \, \epsilon^{\mu\nu\sigma} \, \text{tr} \, U^{-1} \partial_\mu U U^{-1} \partial_\nu U U^{-1} \partial_\sigma U \]
Topologically Massive Yang-Mills Theory

The action is given by

\[ S_{TMYM} = S_{CS} + S_{YM} \]

\[ = -\frac{k}{4\pi} \int_{\Sigma \times [t_i, t_f]} d^3 x \ \epsilon^{\mu\nu\alpha} \ Tr[A_\mu \partial_\nu A_\alpha + \frac{2}{3} A_\mu A_\nu A_\alpha] \]

\[ - \frac{k}{4\pi} \frac{1}{2m} \int_{\Sigma \times [t_i, t_f]} d^3 x \ Tr \ F_{\mu\nu} F^{\mu\nu} \]

Here \( m \) is called the topological mass.

The field equations of this theory are,

\[ \epsilon^{\mu\alpha\beta} F_{\alpha\beta} + \frac{1}{m} D_\nu F^{\mu\nu} = 0 \]
Topologically Massive Yang-Mills Theory

To simplify the notation, we define,

\[ \tilde{A}_\mu = A_\mu + \frac{1}{4m} \epsilon_{\mu \alpha \beta} F^{\alpha \beta} \]

then the momenta are

\[ \Pi^\bar{z} = \frac{ik}{2\pi} \tilde{A}_{\bar{z}} \quad \Pi^\bar{z} = \frac{ik}{2\pi} \tilde{A}_\bar{z} \]

where

\[ \tilde{A}_z = A_z + \frac{i}{4m} F^{0\bar{z}} \quad \tilde{A}_{\bar{z}} = A_{\bar{z}} - \frac{i}{4m} F^{0z} \]

On the wave-functional

\[ A^a_z \Psi = \frac{2\pi}{k} \frac{\delta}{\delta \tilde{A}_z^a} \Psi \quad \tilde{A}^a_z \Psi = \frac{2\pi}{k} \frac{\delta}{\delta A_z^a} \Psi \]
now, we choose the polarization condition that makes the wave-functional

\[ \Psi[A_z, A_\bar{z}, F^{0\bar{z}}, F^{0z}] \rightarrow \Psi[A_\bar{z}, F^{0z}] \]

An infinitesimal gauge transformation on the wave-functional

\[ \delta_\epsilon \Psi[A_\bar{z}, F^{0z}] = \int d^2z \left( \delta_\epsilon A^a_\bar{z} \frac{\delta \Psi}{\delta A^a_\bar{z}} + \delta_\epsilon F^{a0z} \frac{\delta \Psi}{\delta F^{a0z}} \right) \]

\[ = \frac{k}{2\pi} \int d^2z \epsilon^a \left[ D_\bar{z} A^a_z + \frac{i}{4m} \left( D_\bar{z} F^{a0\bar{z}} + D_z F^{a0z} - \partial_z F^{a0z} \right) \right] \Psi \]
from the field equations

\[
\left[ F_{z\bar{z}} - \frac{i}{4m} \left( D_z F^{0\bar{z}} + D_{\bar{z}} F^{0\bar{z}} \right) \right] \Psi = 0
\]

then the infinitesimal gauge transformation becomes

\[
\delta_\epsilon \Psi = \frac{k}{2\pi} \int d^2 z \epsilon^a \left( \partial_{\bar{z}} A_z^a - \frac{i}{4m} \partial_z F^{a0\bar{z}} \right) \Psi
\]

\[
= \frac{k}{2\pi} \int d^2 z \epsilon^a \partial_{\bar{z}} \tilde{A}_z^a \Psi
\]

same solution, using \( \tilde{A}_z = -\partial_{\bar{z}} \tilde{U} \tilde{U}^{-1} \)

\[
\Psi [A_{\bar{z}}, F^{0\bar{z}}] = \exp( k S_{WZW} (\tilde{U}) )
\]
\[ \tilde{A}_z = A_z - \frac{i}{4m} F^{0z} \implies \partial_\zbar \tilde{U} \tilde{U}^{-1} = \partial_z U U^{-1} + \frac{i}{4m} F^{0z} \]

assuming that \( \tilde{U} = U M \)

\[ \implies M = \mathcal{P} \exp \left( \frac{i}{4m} \int F'^{0z} d\zbar \right) \]

where \( F'^{0z} = U^{-1} F^{0z} U \)

Then using Polyakov-Wiegmann identity

\[ \Psi = \exp \left( k S_{ZWZW}(U) + k S_{ZWZW}(M) - \frac{k}{\pi} \int_\Sigma \text{Tr}(U^{-1} \partial_\zbar U \partial_z M M^{-1}) \right) \]
The Equivalence Between TMYM and CS[2]

A redefinition of the gauge field

\[ \hat{A}_\mu = A_\mu + \sum_{n=1}^{\infty} \frac{1}{m^n} \theta^n_\mu (D, F) \]

allows us to write,

\[ S_{CS}(\hat{A}) = S_{CS}(A) + S_{YM}(A) \]

Where D is

\[ D_\mu \bullet = \partial_\mu \bullet + [A_\mu, \bullet] \]

\[ \theta^1_\mu = \frac{1}{4} \epsilon_{\mu\sigma\tau} F^{\sigma\tau} \quad \theta^3_\mu = -\frac{1}{16} \epsilon_{\mu\sigma\tau} D^\sigma D^\rho F^{\rho\tau} + \frac{1}{48} \epsilon_{\mu\sigma\tau} [F^{\sigma\rho}, F^{\tau}_\rho] \]

\[ \theta^2_\mu = \frac{1}{8} D^\sigma F^{\sigma\mu} \]

Equivalence as a Canonical Transformation

\[ \mathcal{A} \rightarrow \mathcal{A} + \delta \Lambda [A] \]

\[ \rightarrow \quad \Psi \rightarrow \Psi' = \exp(i\Lambda) \Psi \]

for large topological mass

\[ \rightarrow \quad \Lambda = -\frac{ik}{\pi m} \int_{\Sigma} Tr(A_{\bar{z}} \theta^1_{\bar{z}}) \]

\[ = \frac{k}{4\pi m} \int_{\Sigma} Tr(A_{\bar{z}} F^{0\bar{z}}) \]

also

\[ \text{det} \left| \frac{\delta \hat{A}}{\delta A} \right| = 1 + \mathcal{O}(1/m^2) \]
The Wilson loop integral is

\[ \hat{W}_{R_i}(C_i) = Tr_{R_i} \left( P \exp i \oint A_\mu dx^\mu \right) \]

A link L is a union of non-intersecting knots \( C_i \)

\[ < \hat{W}_{R_1}(C_1) \ldots \hat{W}_{R_n}(C_n) > \equiv < \hat{W}(L) > \]

\[ < \hat{W}_{R_1}(C_1) \ldots \hat{W}_{R_n}(C_n) > = Z^{-1} \int D\hat{A} e^{-S_{CS}(\hat{A})} \hat{W}_{R_1}(C_1) \ldots \hat{W}_{R_n}(C_n) \]

\[ \beta S_{L+} - \beta^{-1} S_{L-} = z S_{L0} \]

Here, \( S_L \) is a polynomial of \( \beta \) and \( z = z(\beta) \)

The normalization condition is

\[ S_0 = 1 \]
Wilson Loops and Skein Relations

$$\beta < \hat{W}(L_+) > - \beta^{-1} < \hat{W}(L_-) > = z < \hat{W}(L_0)\hat{W}(C_0) > 
\approx Nz < \hat{W}(L_0) >$$

Where

$$\beta = 1 - \frac{2\pi}{k} \frac{1}{2N} + O \left( \frac{1}{k^2} \right)$$

$$z = -i \frac{2\pi}{k} + O \left( \frac{1}{k^2} \right)$$
Wilson Loops and Skein Relations

Using the equivalence

\[ \hat{A}_\mu = A_\mu + \frac{1}{m} \theta^1_\mu + \mathcal{O}(1/m^2) \]

\[ \hat{W}_{R_i}(C_i) = \text{Tr}_{R_i} \left( P \exp i \int \hat{A}_\mu dx^\mu \right) \]

\[ = \text{Tr}_{R_i} \left[ P \left( \exp i \int A_\mu dx^\mu \right) \left( \exp \frac{i}{m} \int \theta^1_\mu dx^\mu \right) \right] + \mathcal{O}(1/m^2) \]

\[ \beta < \hat{W}(L_+) > - \beta^{-1} < \hat{W}(L_-) > \approx z < \hat{W}(L_0)\hat{W}(C_0) > \]

\[ \approx N z < \hat{W}(L_0) > \]
THANK YOU

* The Audience

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