Real-Time AdS/CFT and Applications

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Real-time finite temperature AdS/CFT

AdS/CFT at $T = 0$: Strong-weak duality between a conformal field theory and string theory in a curved (Anti-de Sitter) background.
Trademark example: $\mathcal{N} = 4$ super Yang-Mills (sYM) and IIB string theory on $AdS_5 \times S^5$.

Identification of partition functions: $Z_{sYM} = Z_{IIB_{string}}$

In a simplified set-up, consider the supergravity (sugra) modes on $AdS_5 \times S^5$.

Witten:

$Z_{sYM}[J = \text{source for a BPS opt. } O] = Z_{sugra}[\phi_{sugra}(x^\mu, z) \rightarrow J(x^\mu) \text{ at the AdS bdy}]$

$\Rightarrow$ Correlators of BPS sYM operators can be computed at strong coupling by doing a weakly coupled gravity computation.

Witten diagram:

Witten’s prescription was for Euclidean AdS.

The correlators are then computed in imaginary time. What about real-time correlators?
The problem:

\( AdS_5 \) can be described using different coordinates: global coordinates, which cover the whole space (two time-like boundaries!), or coordinates which make manifest the Poincare symmetry of the field theory (plus a radial coordinate)

\[
ds^2_{AdS} = \frac{1}{z^2}(dx^\mu dx_\mu + dz^2)
\]

and which cover only half of the space.

Which region of AdS must one integrate to get real-time correlators?

What about the different types of real-time correlators? How are they computed from sugra?
To find the answer, we used reverse engineering:
- start from the known expressions of the real-time 3-point correlators

\[
G_F(x_1, x_2, x_3) = (-i)^2 \langle 0 | \mathcal{T}O(x_1)O(x_2)O(x_3) | 0 \rangle
\]
\[
= (-i)^2 \left( \frac{1}{-t_{12}^2 + x_{12}^2 + i\epsilon} \frac{1}{-t_{23}^2 + x_{23}^2 + i\epsilon} \frac{1}{-t_{31}^2 + x_{31}^2 + i\epsilon} \right)^{\Delta/2}
\]
\[
G_{123}(x_1, x_2, x_3) = (-i)^2 \langle 0 | O(x_1)O(x_2)O(x_3) | 0 \rangle
\]
\[
= (-i)^2 \left( \frac{1}{-t_{12}^2 + x_{12}^2 + i\epsilon t_{12}} \frac{1}{-t_{23}^2 + x_{23}^2 + i\epsilon t_{23}} \frac{1}{-t_{31}^2 + x_{31}^2 - i\epsilon t_{31}} \right)^{\Delta/2}
\]
\[
G_R(x_1, x_2, x_3) = \theta(t_{31})\theta(t_{12}) \left( G_{312} - G_{132} + G_{213} - G_{231} \right)
\]
\[
+ \theta(t_{32})\theta(t_{21}) \left( G_{321} - G_{231} + G_{123} - G_{132} \right)
\]

and manipulate until the AdS structure of 3 bulk-to-bdy propagators emerges: the integration needs to be done only over the Poincare patch.
What about the different types of real-time correlators? How are they computed from sugra?

**Our answer**: for Feynman (time-ordered) correlators use Feynman sugra propagators; for retarded/ (causal) correlators use causal sugra propagators. Tantamount to using Veltman's circling rules at $T = 0$ in supergravity [1004.1179].

\[
\begin{align*}
\Delta_F & \quad \rightarrow \\
\Delta^- & \quad \rightarrow \\
\Delta^+ & \quad \rightarrow \\
\Delta_F^* & \quad \rightarrow
\end{align*}
\]

This is the jump-off point for real-time $T \neq 0$ computations from AdS/CFT.
Finite temperature AdS/CFT: the deconfined/high temperature phase of the CFT is holographically dual to AdS with a black hole in it: AdS-Schwarzschild (AdS-S)

\[ ds_{10}^2 = \frac{r^2}{R^2} (-f(r)dt^2 + d\vec{x}^2) + \frac{R^2}{r^2 f(r)} dr^2 + R^2 d\Omega_5 \]

\[ = R^2 \left( -f(z)dt^2 + d\vec{x}^2 + \frac{dz^2}{z^2 f(z)} + d\Omega_5^2 \right), \quad z = \frac{R^2}{r} \]

\[ f(r) = 1 - \frac{r_0^4}{r^4} \]

The Hawking temperature is \( T_H = \frac{r_0}{\pi R^2} \).

AdS-S Penrose diagram

How to integrate over the black hole bulk (given the presence of singularities, horizons)?
Tools of the trade: supergravity bulk-to-boundary propagators, and supergravity vertices.

Consider a massless scalar field \( \phi(p^\mu, u) = F(\omega, \vec{p}, u) \phi_0(p^\mu) \) in the AdS-S background obeys \( \Box F = 0 \):

\[
F''' - \frac{1 + u^2}{u(1 - u^2)} F' + \left( \frac{\omega^2}{u(1 - u^2)^2} - \frac{\vec{p}^2}{u(1 - u^2)} \right) F = 0
\]

where \( \omega, \vec{p} \) and \( u \) are dimensionless quantities

\[
\omega = \frac{E}{2\pi T_H}, \quad \vec{p} = \frac{\vec{P}}{2\pi T_H}, \quad u = \frac{r_0^2}{R^2} z^2.
\]

Furthermore, impose the condition that \( F \) is an incoming wave at the horizon.

- Retarded propagator: \( G_R \equiv F(\omega, \vec{k}, u) \).

- Gibbons & Perry The "Kruskal" vacuum Feynman propagator is the one which exhibits periodicity in imaginary "Schwarzschild" time. Hallmark characteristic of thermal propagators.

Feynman propagator: \( G_F = \text{Re}G_R + i \coth(\omega \pi) \text{Im}G_R \).

\( G_R = G_F - G^- \) etc. \( \Rightarrow \) get the other (Wightman) Green's functions.
What was known since 2002 or how to compute 2-point functions:

**Son & Starinets** conjectured that the retarded 2-point CFT correlator at finite temperature is given by

\[
\langle O(\omega, \vec{k})O(0) \rangle_\beta \propto \sqrt{g} g^{uu} \partial_u F(\omega, \vec{k}, u) \bigg|_{u=0}
\]

based on the zero-temperature limit.

**Son & Herzog** gave a geometric interpretation of the finite temperature Schwinger-Keldysh matrix 2-point correlator by adding sources for the physical and ghost/doubler fields on the two boundaries of the AdS-S Penrose diagram.

**Comments:** Peculiar feature of 2-point functions which arise from \( \int \sqrt{g} \partial \phi \cdot \partial \phi \): when evaluating the quadratic action on-shell, a 2-point function reduces to a boundary term. A genuine integration over the black hole bulk is not needed. Son and Herzog’s prescription was not precise in how the integration over the black hole needs to be carried out.
Our answer: $R - L$ prescription

If one follows Son and Herzog, then

$$S \sim \left( \int_R - \int_L \right) \sqrt{g} \partial \phi \partial \phi,$$

use EOM and keep the R and L boundary terms, with a relative sign contribution [1004.1179].

This gives the 2-point finite temperature Schwinger-Keldysh Green’s function.
Simpler interpretation: the $R - L$ prescription is merely enforcing Veltman's circling rules at finite temperature [1004.1179].

**Kobes, Kobes & Semenoff:** Trade off the real-doubler Schwinger-Keldysh propagator for circling rules diagrams.

Any finite temperature “Feynman” diagram is given by the sum of all diagrams with vertices either circled or uncircled, with the exception of vertices connected to external lines which remain uncircled.

**Largest time equation (identity):** The sum of all diagrams with all vertices either circled or uncircled is 0.

A **retarded n-point function**, with one external vertex having the largest time is given by the sum of all diagrams, with all other vertices being either circled or uncircled.

It is this Green's function computed in real-time formalism that coincides with the analytic continuation of the imaginary time formalism.

The same rules apply to gravity, with the integration over the bulk region only up to horizon. (We need not be concerned with the global black hole space-time.)

The Poincare coordinates are singled out since they are the preferred coordinates in the dual field theory.
Upshot: we give the first concrete formulas for real-time finite-temperature 3-point correlators computed from AdS/CFT [1004.1179]

At finite temperature,

**Retarded 3-point definition**

We define the retarded 3-point correlator to be given by the sum of all diagrams above. After substituting the various $G_{abc}$, the final expression is

$$G_R(q,p,r) \propto \int_0^1 du \sqrt{g} \, F^*(q) F^*(p) F(r)$$

which is consistent with the zero temperature limit result, it is consistent with analytic continuation of the imaginary time Green’s function, and it is manifestly causal.
Applications

Applications of the real-time finite-temperature AdS/CFT formalism:

- study response functions at finite temperature:
  In [1008.4023] we identified a new scale which characterizes the evolution of a high energy jet in the strongly coupled super Yang-Mills plasma: while the maximal distance traveled scales with energy as $E^{1/3}$, most of the jet stops at a smaller scale, proportional to $(EL)^{1/4}$, where $L$ is the size of the region where the jet was created.

- go beyond linear-response (work in progress)

- describe the features of a strongly-coupled CFT’s at finite temperature (extensions of the work of Hofman and Maldacena who, based on a real-time computation of a certain 3-point correlator, at zero temperature, were able to conclude that an isotropic source will not produce back-to-back jets)

- applications to AdS/CM, that is to holographic dualities of gravity and non-relativistic theories with Schrodinger-symmetry [1007.1644] w E. Barnes and C. Wu.
Jet quenching revisited

How far does a localized high-energy excitation travel through the quark-gluon plasma before stopping and thermalizing?

Weakly-coupled plasma: $E^{1/2}$

Strongly-coupled $\mathcal{N} = 4$ super Yang-Mills: $E^{1/3}$

(Maximal distance traveled $\sim (E/\sqrt{\lambda})^{1/3}$ for excitations dual to semi-classical strings. No $\sqrt{\lambda}$-dependence for excitations dual to sugra modes.)

In [1008.4023] we re-opened the problem by posing the question on the field theory side: namely we specify the excitation created on the gauge theory side, and the response (in terms of conserved charge densities) is later measured in the field theory as well. We work at strong coupling and use AdS/CFT duality.

$$\mathcal{L} \rightarrow \mathcal{L} + j^\alpha_\mu A^{\alpha \mu}_{cl},$$

$$A^{\mu}_{cl}(x) = \bar{\epsilon}^\mu \mathcal{N} A \left[ \frac{\tau^+}{2} e^{i \vec{k} \cdot \vec{x}} + \text{h.c.} \right] e^{-\frac{1}{2}(x_0/L)^2} e^{-\frac{1}{2}(x_3/L)^2},$$

$$\vec{k}^\mu = (E, 0, 0, E), \quad E \gg T, EL \gg 1.$$
Analogy: A very high energy $W^+$ boson decaying inside a standard-model quark-gluon plasma and producing high-energy partons moving to the right with net 3rd component of isospin, $\tau^3/2$:

The problem:

The source $A_{cl}^{a\mu}$ creates an excitation that carries energy, momentum, and R charge. We track the R charge density, specifically the large-time behavior ($t \gg$ both $T^{-1}$ and $L$) of

$$\left\langle j^{(3)0}(x) \right\rangle_{A_{cl}}$$

if the system starts in thermal equilibrium at $t = -\infty$. 
This reduces to a retarded 3-point function!

\[ \left\langle j^{(3)\mu}(x) \right\rangle_{A_{cl}} = \frac{1}{2} \int d^4 x_1 \, d^4 x_2 \, G^{(ab3)\alpha\beta\mu}_R(x_1, x_2; x) \, A^a_{\alpha,cl}(x_1) \, A^b_{\beta,cl}(x_2) \]

where

\[ G^{(ab3)\alpha\beta\mu}_R(x_1, x_2; x) = \theta(t - t_1)\theta(t_1 - t_2)\left\langle [[j^{(3)}(x), j^a(x_1)], j^b(x_2)] \right\rangle + \theta(t - t_2)\theta(t_2 - t_1)\left\langle [[j^{(3)}(x), j^b(x_2)], j^a(x_1)] \right\rangle \]

The physical problem of tracking the jet evolution reduces to a technical problem: how to actually compute the 3-point correlators.
Witten diagram for (a) 3-point boundary correlator in imaginary-time AdS$_5$-Schwarzschild and (b) retarded 3-point boundary correlator $G_R(x_1, x_2; x)$ in real-time AdS$_5$-Schwarzschild.

Technical comments (helpful approximations):
- the jet has large energy (WKB approximation useful).
- the R-charge density is measured at scales which are large comparative to $1/E$ or even $1/T$ (one measures a “smeared response”)

Then, the Fourier-transform 3-point correlator factorizes (almost).
Final result:

\[
(\partial_t - \frac{1}{2\pi T} \nabla^2) \langle j^{(3)0}(x) \rangle_{A_{c1}} \sim \tilde{Q}^{(3)} \Theta(x),
\]

where the charge deposition function is

\[
\Theta(x) \simeq 2 \delta_L(x^-) \theta(x^+) \left\{ \begin{array}{l}
\frac{(4c^4EL)^2}{(2\pi T)^8(x^+)^9} \Psi\left(-\frac{c^4EL}{(2\pi T x^+)^4}\right), \\
\frac{(2\pi T)^4}{2(c_2 L)^2} \frac{E}{E_{1/3}^1} \Psi(0) \exp\left(-\frac{c_1(2\pi T)^{4/3} x^+}{E_{1/3}^1}\right),
\end{array} \right.
\]

with \( \Psi(y) = e^{-2y^2} \), \( c \equiv \frac{r^2(1/4)}{(2\pi)^{1/2}} \), \( c_1 \simeq 0.927 \), \( c_2 \simeq 3.2 \).

\[\text{charge deposited} \]

\[x^+ \]

\[\sim (x^+)^{-9}\]

\[\text{exponential fall-off}\]

\[E_1^{1/3} T^{-4/3} \quad (EL)^{1/4} T^{-1}\]

\[x^+\]

\[\text{exponential fall-off}\]

\[E_1^{1/3} T^{-4/3} \quad (EL)^{1/4} T^{-1}\]
Summary

The response to the high energy jet was computed by measuring the R-charge in its wake at late times. This reduces to a real-time finite temperature 3-point retarded \( \langle j_{\text{source}}^\dagger j_{\text{response}} j_{\text{source}} \rangle \) correlator.

Our previous work on how to set up the computation of such correlators on the gravity side has enabled us proceed.

The conclusion is that while the \( E^{1/3} \) scale is still present, as the maximal distance the jet travels (if \( T = 1 \)), the jet deposits most of its charge at the smaller distance \( (EL)^{1/4} \).

Other dimensions scales: For \( AdS_{d+1} \),

- the maximal distance traveled scales as \( E^{(d-2)/(d+2)} \)
- the smaller scale, where most of the charge is deposited scales as \( (EL)^{(d-2)/(2d)} \)

To appear: There is a rather simple interpretation for the new scale, as well as for the power law fall-off \( x^{-9} \) which generalizes to \( x^{3-4\Delta} \).

Work in progress: Correlator of two R-charge densities at late times, in the background of the source. Consider finite temperature and finite chemical potential.

Still more to do: compute the energy density at late times; higher n-point correlators; stringy (\( \alpha' \)) corrections; beyond linear response.