Cosmological brane solutions in supergravity

Kunihito Uzawa

Department of Physics, Kindai University

(K. Maeda, N. Ohta, K. Uzawa, JHEP 06 (2009) 051)
(M. Minamitsuji, K. Uzawa, arXiv 1011.2376 [hep-th])
[1] Introduction

- The dynamical solutions of supergravity have a number of important applications: For example, dynamics of moduli, analysis of the early universe, BH in expanding universe.

- For the general p-brane system, that the structure of warp factor which depends on the space and time is different from the usual “product type” ansatz.

- More general dynamical brane solutions arise if the gravity is coupled not only to single gauge field but to several combinations of scalars and forms as intersecting brane solutions in the supergravity.

- We give dynamical solutions of p-brane systems in D-dimensional theories, which may have more general applications to cosmology.
Dynamical brane solutions:

**Single p-brane**
(Gibbons, Lu, Pope, Phys.Rev.Lett.94:131602,2005)
(H.Kodama; K.Uzawa, JHEP 0507:061,2005)
(H.Kodama; K.Uzawa, JHEP 0603:053,2006)

**Intersecting branes**
(K. Maeda, N. Ohta, K. Uzawa, JHEP 0906:051,2009)
(M. Minamitsuji, K. Uzawa, arXiv 1011.2376 [hep-th])

**p-branes with cosmological constant**
(K. Maeda, M. Minamitsuji, N. Ohta, K. Uzawa,
(M. Minamitsuji, K. Uzawa, arXiv 1011.2376 [hep-th])

Cosmology

No

Yes
★ Kaluza-Klein compactification

"product type" ansatz

\[ ds^2 = a(x, y)q_{\mu\nu}dx^\mu dx^\nu + b(x, y)u_{ij}dy^i dy^j, \]

\[ a(x, y) = a_0(x)a_1(y), \quad b(x, y) = b_0(x)b_1(y) \]

⇒ The scale \( a(x, y) \) is written by the product of the function \( a_0(b_0) \) and \( a_1(b_1) \).

☆ Dynamical p-brane warped compactification

Not "product type" ansatz

\[ ds^2 = h^a(x, y)q_{\mu\nu}dx^\mu dx^\nu + h^b(x, y)u_{ij}dy^i dy^j, \]

\[ h(x, y) = h_0(x) + h_1(y) \]

⇒ The scale \( h(x, y) \) is written by the linear combination of the function \( h_0 \) and \( h_1 \).

• expansion law: \( a(\tau) \propto \tau^{\lambda}, \quad \lambda = (p+1)/(D+p-1) < 1 \) for \( D > 2 \)
Dynamical solution of p-brane system

(H. Kodama & K. Uzawa ; JHEP 07 (2005) 061)

\[ ds^2 = h^{-\frac{(D-p-3)}{(D-2)}} \eta_{\mu \nu} dx^\mu dx^\nu + h^{\frac{(p+1)}{(D-2)}} (dr^2 + r^2 d\Omega_{D-p-2}^2), \]
\[ h(t, r) = c_1 t + c_2 + Mr^{-D+p+3}, \quad F_{p+2} = dh^{-1} \wedge dt \wedge dx^1 \wedge \cdots \wedge dx^p, \]
\[ e^\phi = h^{c/2}, \quad c^2 = 4 - 2(p+1)(D-p-3)(D-2)^{-1} \]

- 10-dim D3-brane solution

\[ ds^2 = (c_1 t + c_2)^{-1/2} \eta_{\mu \nu} dx^\mu dx^\nu + (c_1 t + c_2)^{1/2} (dr^2 + r^2 d\Omega_5^2) \]

Kasner solution

\[ r \to \infty, \]
\[ ds^2 = (\frac{r}{M})^2 \eta_{\mu \nu} dx^\mu dx^\nu + (\frac{M}{r})^2 dr^2 + d\Omega_5^2 \]

\[ r \to 0 \]

AdS$_5 \times $ S$_5$
Dynamical solution of intersecting brane system
(K. Maeda, N. Ohta, K. Uzawa, JHEP 06 (2009) 051)

\[ S = \frac{1}{2\kappa^2} \int \left[ R \ast 1 - \frac{1}{2} d\phi \wedge \ast d\phi - \frac{1}{2} \frac{1}{(p_r + 2)!} \epsilon^{\epsilon_r c_r \phi} F_{(p_r+2)} \wedge \ast F_{(p_r+2)} ight] \]

\[ - \frac{1}{2} \frac{1}{(p_s + 2)!} \epsilon^{\epsilon_s c_s \phi} F_{(p_s+2)} \wedge \ast F_{(p_s+2)} ] , \]

\( R \): Ricci scalar, \( \phi \): scalar field, \( F_{(p_l+2)} \): \((p_l+2)\)-form
\( \kappa \): gravitational constant

\( c_l \) and \( \epsilon_l \) \((l=r, s)\) are constants given by

\[ c_l^2 = 4 - \frac{2(p_l + 1)(D - p_l - 3)}{D - 2} , \]

\[ \epsilon_l = \begin{cases} + & \text{if } p_l - \text{brane is electric} \\ - & \text{if } p_l - \text{brane is magnetic} \end{cases} \]
\[ d s^2 = h_r^{\alpha-1} h_s^{\beta-1} \left[ q_{\mu\nu}(X) dx^\mu dx^\nu + h_r \gamma_{ij}(Y_1) dy^i dy^j 
+ h_s w_{mn}(Y_2) dv^m dv^n + h_r h_s u_{ab}(Z) dz^a dz^b \right] \]

- **D-dimensional metric**: \( \alpha = (p_r + 1)(D - 2)^{-1}, \quad \beta = (p_s + 1)(D - 2)^{-1} \)

- **(p+1)-dim world volume**: (p_s-p)-dim relative transverse
- **(p_r-p)-dim relative transverse**: (D-p_r-p_s-1)-dim transverse

- world-volume: Both branes are extended.
- relative transverse : Only one of the two branes is extended.
- transverse : Both branes are not extended.

- \( \alpha \) and \( \beta \) are constants.

| \( x^0 \) | \( x^1 \) | \ldots | \( x^p \) | \( y^1 \) | \ldots | \( y^{p_s-p} \) | \( v^1 \) | \ldots | \( v^{p_r-p} \) | \( z^1 \) | \ldots | \( z^{D+p_r-p_s-1} \) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Dp_r | o | o | o | o | o | o | o | o |
| Dp_s | o | o | o | o | o | o | o | | | | |

- world volume
- relative transverse
- transverse
The field equations of intersecting branes allow for the following three kinds of possibilities on $p_r$- and $p_s$-branes in D dimensions

Case I: Both $h_r$ and $h_s$ depend on the overall transverse coordinates:

(K. Maeda, N. Ohta, K. Uzawa, JHEP 0906:051, 2009)

\[ h_r = h_r(x, z), \quad h_s = h_s(x, z) \]

Case II (new solution): Only $h_s$ depends on the overall transverse coordinates, but the other $h_r$ does on the corresponding relative coordinates:

\[ h_r = h_r(x, y), \quad h_s = h_s(x, z) \]

Case III (new solution): Each of $h_r$ and $h_s$ depends on the corresponding relative coordinates:

\[ h_r = h_r(x, y), \quad h_s = h_s(x, \nu) \]

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Case I (K. Maeda, N. Ohta, K. Uzawa, JHEP 06 (2009) 051)

\[ e^\phi = h_r^{c_r c_r/2} h_s^{c_s c_s/2}, \]
\[ F_{(p_r+2)} = d \left[ h_r^{-1}(x, z) \right] \wedge \Omega(X) \wedge \Omega(Y_2), \]
\[ F_{(p_s+2)} = d \left[ h_s^{-1}(x, z) \right] \wedge \Omega(X) \wedge \Omega(Y_1), \]

\[ \Omega(X), \Omega(Y_1), \Omega(Y_2) \) are volume form of the \( X, Y_1, Y_2 \) space.

Field equations can be satisfied only if there is only one function depending on both transverse space coordinates and time.

For \( u_{ij} = \delta_{ij} \) and \( h_r = h_r(x, z) \), the solution can be obtained explicitly as
\[ h_r(x, z) = A_\mu x^\mu + B + \sum_l \frac{M_l}{|z^a - \bar{z}_l^a|^{D+p-p_r-p_s-3}}, \]
\[ h_s(z) = C + \sum_c \frac{M_c}{|z^a - \bar{z}_c^a|^{D+p-p_r-p_s-3}}. \]

\( A_\mu, B, C, M_l, M_c \) are constants

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In order to satisfy the field equations, two branes have to obey the intersection rules

☆ Intersections rules involving the two M-branes

\[ \bar{p} + 1 = \frac{1}{9}(p_r + 1)(p_s + 1), \quad p_r \cap p_s = \bar{p} \]

⇒ \[ M2 \cap M2=0, \quad M5 \cap M5=3, \quad M2 \cap M5=1 \]

☆ Intersections rules involving the two D-branes

\[ \bar{p} = \frac{1}{2}(p_r + p_s - 4) \]

⇒ \[ Dp \cap Dp=p-2, \quad D(p-2) \cap Dp=p-3, \quad D(p-4) \cap Dp=p-4 \]

\[ F1 \cap NS5=1, \quad NS5 \cap NS5=3 \]

\[ F1 \cap Dp=0, \quad Dp \cap NS5=p-1, \quad (1 \leq p \leq 6) \]

• Ansatz:

\[ e^\phi = h_r^{\varepsilon_r c_r/2} h_s^{\varepsilon_s c_s/2}, \]

\[ F_{(p_r+2)} = d [h_r^{-1}(x, y)] \wedge \Omega(X) \wedge \Omega(Y_2), \]

\[ F_{(p_s+2)} = d [h_s^{-1}(x, z)] \wedge \Omega(X) \wedge \Omega(Y_1), \]

• \( \Omega(X), \Omega(Y_1), \Omega(Y_2) \) are volume form of the \( X, Y_1, Y_2 \) space.

Field equations can be satisfied only if there is only one function depending on both transverse space coordinates and time.

For \( u_{i,j} = \delta_{i,j} \) and \( h_r = h_r(x, y) \), the solution can be obtained explicitly as

\[ h_r(x, y) = A_\mu x^\mu + B + \sum_l \frac{M_l}{|y^i - y_l^i|^{p_s-p-2}}, \]

\[ h_s(z) = C + \sum_c \frac{M_c}{|z^a - z_c^a|^{D+p-p_r-p_s-3}}, \]

Intersections rules involving the two D-branes and M-branes are the same as in the case I.

- Ansatz:
  \[ e^{\phi} = h_r^{c_r c_r} h_s^{c_s c_s} / 2, \]
  \[ F_{(p_r+2)} = h_s d \left[ h_r^{-1}(x, y) \right] \wedge \Omega(X) \wedge \Omega(Y_2), \]
  \[ F_{(p_s+2)} = h_r d \left[ h_s^{-1}(x, v) \right] \wedge \Omega(X) \wedge \Omega(Y_1), \]

- \( \Omega(X), \Omega(Y_1), \Omega(Y_2) \) are volume form of the \( X, Y_1, Y_2 \) space.

Field equations can be satisfied only if there is only one function depending on both transverse space coordinates and time.

For \( u_{ij} = \delta_{ij} \) and \( h_r = h_r(x, y) \), the solution can be obtained explicitly as

\[
\begin{align*}
  h_r(x, y) &= A_{\mu} x^{\mu} + B + \sum_l \frac{M_l}{|y^i - y_l|^p - p - 2}, \\
  h_s(v) &= C + \sum_c \frac{M_c}{|v^m - v_c|^p - p - 2},
\end{align*}
\]

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Intersections rules involving the two M-branes
(M. Minamitsuji, N. Ohta, K. Uzawa, To be appeared in PRD, arXiv 1007.1762 [hep-th])

\[
\bar{p} + 3 = \frac{1}{9} (p_r + 1)(p_s + 1), \quad p_r \cap p_s = \bar{p}
\]

⇒ M5 ∩ M5=1

Intersections rules involving the two D-branes
(M. Minamitsuji, N. Ohta, K. Uzawa, To be appeared in PRD, arXiv 1007.1762 [hep-th])

\[
\bar{p} = \frac{1}{2} (p_r + p_s - 8)
\]

⇒ Dp ∩ Dp=p-4,
   D(p-2) ∩ Dp=p-5,
   D(p-4) ∩ Dp=p-6, 
   D(p-6) ∩ Dp=p-7, 
   D(p-8) ∩ Dp=p-8, 
   NS5 ∩ NS5=1
   Dp ∩ NS5=p-3,   (3 ≤ p ≤ 8)
★ Cosmology:

We assume an isotropic and homogeneous three space in the four-dimensional spacetime. Note that the time dependence in the metric comes from only one brane in the intersections.

- Solutions in the original higher-dimensional theory (10D or 11D).

- For each case, the scale factor of 4-dimensional universe is given by $a(\tau) \propto \tau^\lambda$, where $\tau$ denotes the cosmic time.

- Since the three-dimensional spatial space of our universe stays in the transverse space to the brane, D-dimensional theory gives the fastest expansion of our universe.

- The power of the scale factor becomes $\lambda = (p+1)/(D+p-1) < 1$ for $D>2$. It is impossible to find the cosmological model that our universe is accelerating expansion.
The fastest expanding case for each brane system:

1. M-brane: M2-M5 for case I in 11D theory \( \Rightarrow \lambda = \frac{2}{5} \)

2. D-brane: D2-D6 and D4-D6 (case I, II, III) and D6-D6 (case I & II) in 10D theory \( \Rightarrow \lambda = \frac{7}{15} \)

3. F1 string: F1-D6 for case I & II in 10D theory \( \Rightarrow \lambda = \frac{7}{15} \)

4. NS5-brane: NS5-D6 for case I & II in 10D theory \( \Rightarrow \lambda = \frac{7}{15} \)

For the fastest expanding case, since the power is small, it cannot give a realistic expansion law such as that in the matter dominated era \( a(\tau) \propto \tau^{2/3} \) or that in the radiation dominated era \( a(\tau) \propto \tau^{1/2} \). Hence we conclude that in order to find a realistic expansion of the universe in this type of models, one have to include additional matter fields.

\[ \lambda = \frac{(p+1)}{(D+p-1)}, \text{ (higher dimension)} \]
★ Single brane with cosmological constant:

\[
S = \frac{1}{2\kappa^2} \int \left[ (R - 2e^{\alpha\phi}\Lambda) \ast 1_D - \frac{1}{2} d\phi \wedge *d\phi - \frac{1}{2 \cdot (p+2)!} e^{\epsilon\phi} F_{(p+2)} \wedge *F_{(p+2)} \right],
\]

\[
c^2 = N - \frac{2(p+1)(D-p-3)}{D-2}, \quad \epsilon = \pm 1, \quad \alpha = \left[ -N + \frac{2(D-p-3)}{D-2} \right] (\varepsilon c)^{-1}
\]

Solution:

\[
ds^2 = h^a(x, z)\eta_{\mu\nu}(X)dx^\mu dx^\nu + h^b(x, z)\delta_{ij}(Z)dz^i dz^j,
\]

\[
a = -\frac{4(D-p-3)}{N(D-2)}, \quad b = \frac{4(p+1)}{N(D-2)},
\]

\[
e^\phi = h^{2\epsilon c/N}, \quad F_{(p+2)} = \frac{2}{\sqrt{N}} d(h^{-1}) \wedge \sqrt{-q} dx^0 \wedge dx^1 \wedge \cdots \wedge dx^p
\]

\[
h(x, z) = h_0(x) + h_1(z)
\]
\[ c^2 = N - \frac{2(p + 1)(D - p - 3)}{D - 2}, \quad \epsilon = \pm 1, \quad \alpha = \left[ -N + \frac{2(D - p - 3)}{D - 2} \right] (\epsilon c)^{-1} \]

- Scalar field: trivial (c\neq 0) --- 6D Salam & Sezgin model
  \( \Rightarrow \) Asymptotic Milne universe: \( a(\tau) \propto \tau \)

- Scalar field: trivial (c=0) --- Maki & Shiraishi model
  \( \Rightarrow \) Asymptotic de Sitter universe
  (M. Minamitsuji, K. Uzawa, arXiv 1011.2376 [hep-th])
[3] Constraints of warped compactification:

D-dimensional spacetime

$$ds^2 = A^2(y) \left[ q_{\mu\nu}(X) dx^\mu dx^\nu + u_{ij}(Y) dy^i dy^j \right]$$

\[ \text{d}_1\text{-dimensional spacetime} \quad \text{(D-d}_1\text{-)dimensional internal space} \]

- Fields with positive kinetic terms
- No higher curvature correction in the background
- \(d_1\)-dimensional effective Newton constant is finite.

$$\frac{1}{\kappa^2} \sim \int_Y A^{D-2} 1_Y$$
Einstein equation:

\[
\frac{d_1}{(D-2)} A^{D-2} \triangle_Y A^{D-2} = A^{2(D-2)} R(X) + A^{2(D-1)} \bar{T},
\]

\[
\bar{T} \equiv -T^{\mu\mu} + \frac{d_1}{D-2} T,
\]

Integrating over the compact internal space by parts:

\[
-\frac{d_1}{D-2} \int_Y d^{(D-d_1)} y \sqrt{u} u^{ij} \partial_i A^{D-2} \partial_j A^{D-2} = \int_Y d^{(D-d_1)} y \sqrt{u} [R(X) + A^2 \bar{T}]
\]

de Sitter compactification may be allowed if \( \bar{T} < 0 \)
Contributions to $\bar{T}$

(1) Potential term

$$T_{MN} = -V g_{MN}, \quad \rightarrow \quad \bar{T} = -\frac{2d_1}{D - 2} V$$

| $V > 0$, | $\rightarrow$ | $\bar{T} < 0$, |
| $V < 0$, | $\rightarrow$ | $\bar{T} > 0$ |

(2) Field strength

$$T_{MN} = \frac{1}{2 \cdot n!} \left( n F_{MP_2 \cdots P_n} F_{N P_2 \cdots P_n} - \frac{1}{2} g_{MN} F^2 \right),$$

$$\rightarrow \quad \bar{T} = \frac{1}{2 \cdot (n - 1)!} \left[ -F_{\mu P_1 \cdots P_{n-1}} F^{\mu P_1 \cdots P_{n-1}} + \frac{d(n - 1)}{(D - 2)n} F^2 \right]$$

If the field strength has only components along the $(D-d_1)$-dimensional space, $\bar{T} \geq 0$
The trace over the index $\mu$ can be written by a particular order of contractions of the indices

$$F_{\mu P_1 \ldots P_{n-1}} F_{\nu P_1 \ldots P_{n-1}} = \frac{d_1}{n} F^2,$$

$$\Rightarrow \quad \mathcal{T} = - \frac{d(D - n - 1)}{2(D - 2) \cdot n!} F^2 \geq 0$$

For $D=n+1 \Rightarrow$ Minkowski spacetime

de Sitter compactification may be allowed if the potential is positive.

※ The positive potential does not always lead to de Sitter compactification.
D-dimensional spacetime

\[ ds^2 = A^2(x, y) \left[ q_{\mu\nu}(X) dx^\mu dx^\nu + u_{i,j}(Y) dy^i dy^j \right] \]

Einstein equation:

\[
\frac{(D + d_1 - 2)}{\alpha} A^{-\alpha} \Delta_X A^\alpha + \frac{d_1}{(D - 2)A^{D-2}} \triangle_Y A^{D-2} = R(X) + A^2 \bar{T},
\]

\[ \alpha = \frac{(D - 2)(d_1 - 1)}{D + d_1 - 2}, \]

We can not abandon the possibility \( R(X) \geq 0 \) if the function \( A \) satisfies

\[
\frac{(D + d_1 - 2)}{\alpha} A^{-\alpha} \Delta_X A^\alpha \geq 0
\]
[4] Summary:

☆ We give some dynamical intersecting brane solutions in ten- or eleven-dimensional supergravity.

• The cosmological solutions we found have the property that they are genuinely higher-dimensional in the sense that one can never neglect the dependence on the coordinates of transverse space.

☆ We found that these solutions give FLRW universe if we regard the homogeneous and isotropic as our spacetime.

• The power of the scale factor is so small that the solutions of field equations cannot give a realistic expansion law. This means that we have to consider additional matter in order to get a realistic expanding universe.

• de Sitter solution can be found in the p-brane with cc.