INTRINSICALLY QUANTUM-MECHANICAL GRAVITY
AND THE COSMOLOGICAL CONSTANT PROBLEM

Philip D. Mannheim
University of Connecticut
Seminar in Mexico City
November 2010
INTRINSICALLY QUANTUM-MECHANICAL GRAVITY
AND THE COSMOLOGICAL CONSTANT PROBLEM

Philip D. Mannheim
University of Connecticut

Seminar in Montreal
November 2010
GHOST PROBLEM AND UNITARITY
2. P. D. Mannheim, Conformal gravity challenges string theory, Pascos-07, July 2007 (0707.2283 [hep-th]).

PT QUANTUM MECHANICS

PAIS-UHLENBECK FOURTH-ORDER OSCILLATOR

CONFORMAL GRAVITY AND THE COSMOLOGICAL CONSTANT PROBLEM
1 THE NEED FOR QUANTUM GRAVITY

WHAT IS STATUS OF

$$- \frac{1}{8\pi G} G_{\mu\nu} = T_{\mu\nu}. \quad (1)$$

IS IT A C-NUMBER OR A Q-NUMBER EQUATION? Q-NUMBER MAKES NO SENSE SINCE EINSTEIN GRAVITY NOT RENORMALIZABLE. BUT EVEN IF ONLY USE MACROSCOPICALLY, $T_{\mu\nu}$ IS STILL QUANTUM-MECHANICAL SINCE CMB AND WHITE DWARF STARS EXIST.

SO MACROSCOPICALLY TREAT AS THE SEMI-CLASSICAL EQUATION IN STATE $|Q\rangle$ WITH CLASSICAL $G_{\mu\nu}$:

$$- \frac{1}{8\pi G} G_{\mu\nu} = \langle Q | T_{\mu\nu} | Q \rangle \quad (2)$$

HOWEVER $G_{\mu\nu}$ WILL DEPEND ON $\hbar$:

$$\rho_{\text{blackbody}} = \pi^2 (k_B T)^4 / 15 (\hbar c)^3, \quad M_{\text{CHANDRASEKHAR}} \sim (\hbar c / G)^{3/2} / m_p^2.$$  

WORSE: ALL QUANTIZED FIELDS HAVE AN INFINITE ZERO-POINT ENERGY DENSITY (DIRAC).  

SO SUBTRACT OFF BY NORMAL-ORDERING OR DIMENSIONAL REGULARIZATION

$$- \frac{1}{8\pi G} G_{\mu\nu} = \langle Q | T_{\mu\nu} | Q \rangle - \langle \Omega | T_{\mu\nu} | \Omega \rangle. \quad (3)$$

NOT JUSTIFIABLE SINCE GRAVITY COUPLES TO ENERGY NOT ENERGY DIFFERENCE.

SINCE $H = \sum (a^\dagger a + 1/2)\hbar \omega$, CANNOT CONSISTENTLY USE POSITIVE ENERGY BLACK-BODY PHOTONS FOR COSMOLOGY OR FILLED POSITIVE ENERGY FERMI SEA FOR WHITE DWARF STARS WITHOUT ALSO INCLUDING ZERO-POINT CONTRIBUTION. AND YET THIS IS HOW STANDARD GRAVITY IS COMMONLY USED.
TWO COMMONLY USED THEOREMS THAT SUGGEST CAN IGNORE ZERO-POINT PROBLEM:
(1) IF VACUUM LORENTZ INVARIANT \( E' = \gamma (E + \vec{v} \cdot \vec{P}) = \gamma E = E \), TO THUS OBTAIN \( E = 0 \).
(2) IF VACUUM LORENTZ INVARIANT, \( \langle \Omega | T_{\mu\nu} | \Omega \rangle = -\Lambda g_{\mu\nu} \).

FOR INSTANCE, IN CONSTANT SCALAR FIELD CONFIGURATION WITH \( \phi = \phi_0 \) OBTAIN \( T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} V(\phi) = -g_{\mu\nu} V(\phi_0) \), TO GIVE ENERGY DENSITY \( T_{00} = \Lambda = V(\phi_0) \). HOWEVER, FOR THIS CASE TOTAL \( E = \int d^3 x T_{00} = \infty \), AND THUS \( E \) IS NOT EQUAL TO ZERO.

HOWEVER, STILL CONSISTENT SINCE \( E = \gamma E \) ALSO HAS A SECOND SOLUTION: \( E = \infty \). THUS \( E = 0 \) NOT REQUIRED AFTER ALL.

NOW ZERO-POINT ENERGY IS ALSO INFINITE, BUT SO IS ZERO-POINT ENERGY DENSITY, SO ZERO-POINT ENERGY DENSITY CANNOT BE DESCRIBED AS HAVING FINITE \( \Lambda \). BUT WORSE......

ALL QUANTIZED FIELDS HAVE A ZERO-POINT PRESSURE \( p \), AND FOR ALL QUANTIZED FIELDS \( \rho \) AND \( p \) HAVE SAME SIGN, AND THUS DO NOT ACT AS A COSMOLOGICAL CONSTANT AT ALL.

FOR FREE QUANTIZED FERMION FIELDS WITH \( k^\mu = ((k^2+m^2)^{1/2}, \vec{k}) \) WE GET MANIFEST LORENTZ TENSOR

\[
\langle \Omega | T^{\mu\nu}_M | \Omega \rangle = -\frac{2\hbar}{(2\pi)^3} \int_{-\infty}^{\infty} d^3 k \frac{k^\mu k^\nu}{k^0}. \tag{4}
\]

\[
\langle \Omega | T^{\mu\nu}_M | \Omega \rangle = (\rho_M + p_M) U^\mu U^\nu + p \eta^{\mu\nu}, \quad \eta^{\mu\nu} \langle \Omega | T^{\mu\nu}_M | \Omega \rangle = 3p_M - \rho_M, \tag{5}
\]

\[
\rho_M = -\frac{\hbar}{4\pi^2} \left( K^4 + m^2 K^2 - \frac{m^4}{4} \ln \left( \frac{4K^2}{m^2} \right) + \frac{m^4}{8} \right),
\]

\[
p_M = -\frac{\hbar}{12\pi^2} \left( K^4 - m^2 K^2 + \frac{3m^4}{4} \ln \left( \frac{4K^2}{m^2} \right) - \frac{7m^4}{8} \right). \tag{6}
\]
AND FOR MASSLESS QUANTIZED FIELDS $\rho = 3p$ SINCE $k^\mu k_\mu = 0$.

ZERO-POINT IS DIFFERENT FROM A COSMOLOGICAL CONSTANT AS IT HAS $T^{\mu\nu} = -\Lambda g^{\mu\nu}$

FOR ZERO-POINT FLUCTUATIONS THERE IS AN EXTRA TIMELIKE REFERENCE VECTOR $U^\mu$ SINCE FOR EVERY MODE $k^\mu$ WE CAN WRITE $k^\mu = (k^2 + m^2)^{1/2} U^\mu + (0, \vec{k})$. VACUUM NOT STRUCTURELESS OR EMPTY – RATHER IT IS FULL TO THE BRIM WITH MODES. THUS NO VIOLATION OF LORENTZ INVARIANCE, AND THE TWO THEOREMS DO NOT HOLD.

IF REGULATE ZERO-POINT ENERGY MOMENTUM TENSOR BY PAULI-VILLARS OBTAIN

$$\rho_{\text{REG}} = -p_{\text{REG}} = -\frac{\hbar}{16\pi^2} \left( m^4 \ln m^2 + \sum \eta_i M_i^4 \ln M_i^2 \right).$$ (7)

SO WHEN FINITE, VACUUM ENERGY DENSITY DOES INDEED BEHAVE LIKE A COSMOLOGICAL CONSTANT. HOWEVER VERY BIG IF REGULATOR MASSES ARE BIG.

ADDITIONALLY, A FURTHER CONTRIBUTION TO COSMOLOGICAL CONSTANT IS INDUCED WHEN GENERATE MASS BY SYMMETRY BREAKING AS GO THROUGH ELECTROWEAK PHASE TRANSITION.

TWO VACUUM PROBLEMS (ZERO-POINT AND $\Lambda$) NOT ONE – MUST SOLVE THEM SIMULTANEOUSLY.

BOTH PROBLEMS OCCUR IN FLAT SPACE IN THE ABSENCE OF GRAVITY ——— SO HOW IS GRAVITY TO SOLVE THEM?
NEED A SYMMETRY TO CANCEL EVERYTHING. TWO KNOWN: SUPERSYMMETRY AND CONFORMAL SYMMETRY. HOWEVER, SUPERSYMMETRY FAILS TO EFFECT CANCELLATION ONCE SUPERSYMMETRY IS BROKEN, WHEREAS CONFORMAL SYMMETRY CONTINUES TO DO SO.

\[ T_{\text{MAT}}^{\mu\nu} = (\rho_{\text{MAT}} + p_{\text{MAT}})U^\mu U^\nu + p_{\text{MAT}}g^{\mu\nu} - \Lambda_{\text{MAT}}g^{\mu\nu}, \quad g_{\mu\nu}T_{\text{MAT}}^{\mu\nu} = 3p_{\text{MAT}} - \rho_{\text{MAT}} - 4\Lambda_{\text{MAT}} = 0 \]

CONFORMAL INVARIANCE: \( I_W = -\alpha_g \int d^4x (-g)^{1/2}C_{\lambda\mu\nu\rho}C^{\lambda\mu\nu\rho} \) WITH DIMENSIONLESS \( \alpha_g \).

GRAVITY NOW RENORMALIZABLE AND UNITARY (BENDER AND MANNHEIM)

MASSLESS GRAVITON HAS ZERO-POINT ENERGY-DENSITY AND PRESSURE. AND SINCE THE GRAVITATIONAL EQUATION OF MOTION

\[ T_{\text{UNIV}}^{\mu\nu} = -4\alpha_g W^{\mu\nu} + T_{\text{MAT}}^{\mu\nu} = T_{\text{GRAV}}^{\mu\nu} + T_{\text{MAT}}^{\mu\nu} = 0 \]

SURVIVES RENORMALIZATION, GRAVITON ZERO-POINT CANCELS MATTER ZERO-POINT IDENTICALLY, AND DOES SO EVEN AFTER SYMMETRY IS BROKEN AND MASS IS GENERATED.
GRAVITY HAS TO KNOW ABOUT MATTER QUANTUM STRUCTURE TO EFFECT CANCELLATION

CURVATURE HAS TO BE ENTIRELY DUE TO QUANTIZATION

IN ABSENCE OF QUANTIZATION GEOMETRY IS MINKOWSKI

IN CONFORMAL INVARIANT THEORY THIS MUST BE SO SINCE NO INTRINSIC CURVATURE LENGTH SCALES IN A CONFORMAL CLASSICAL ACTION

GRAVITY IS QUANTIZED BY ITS COUPLING TO QUANTIZED MATTER FIELDS, AND IF MATTER ACTION IS CONFORMAL INVARIANT, GRAVITY GETS ITS MASS SCALES ENTIRELY FROM QUANTUM MATTER FIELD DYNAMICS

EXPAND GRAVITY AS A POWER SERIES IN $\hbar$, NOT AS A POWER SERIES IN $G$
ALL YOU NEED TO KNOW ABOUT CONFORMAL GRAVITY

\[ g_{\mu\nu} \rightarrow e^{2\alpha(x)} g_{\mu\nu}, \quad C_{\lambda\mu\nu\kappa} \rightarrow e^{2\alpha(x)} C_{\lambda\mu\nu\kappa} \]  

(8)

\[ ds^2 \rightarrow e^{2\alpha(x)} ds^2, \quad ds^2 = 0 \rightarrow ds^2 = 0 \]  

(9)

\[ C_{\lambda\mu\nu\kappa} = R_{\lambda\mu\nu\kappa} + \frac{1}{6} R_\alpha^\alpha \left[ g_{\lambda\nu} g_{\mu\kappa} - g_{\lambda\kappa} g_{\mu\nu} \right] - \frac{1}{2} \left[ g_{\lambda\nu} R_{\mu\kappa} - g_{\lambda\kappa} R_{\mu\nu} - g_{\mu\nu} R_{\lambda\kappa} + g_{\mu\kappa} R_{\lambda\nu} \right] \]  

(10)

\[ I_W = -\alpha g \int d^4 x (-g)^{1/2} C_{\lambda\mu\nu\kappa} C^{\lambda\mu\nu\kappa} = -2\alpha g \int d^4 x (-g)^{1/2} \left[ R^{\mu\nu} R_{\mu\nu} - \frac{1}{3} (R_\alpha^\alpha)^2 \right] \]  

(11)

\[ -4\alpha g W^{\mu\nu} + T^{\mu\nu} = 0 \]  

(12)

\[ W^{\mu\nu} = \frac{1}{2} g^{\mu\nu} (R_\alpha^\alpha)^{;\beta \beta} + R^{\mu\nu;\beta \beta} - R^{\mu;\beta \beta \nu} - R^{\nu;\beta \beta \mu} - 2 R^{\mu\beta} R_{\beta}^{\nu} + \frac{1}{2} g^{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} - \frac{2}{3} g^{\mu\nu} (R_\alpha^\alpha)^{;\beta \beta} \]  

(13)

\[ + \frac{2}{3} (R_\alpha^\alpha)^{;\mu\nu} + \frac{2}{3} R_\alpha^\alpha R^{\mu\nu} - \frac{1}{6} g^{\mu\nu} (R_\alpha^\alpha)^2 \]  

Outside static source :  

\[ g_{00} = \frac{1}{g_{rr}} = 1 - \frac{2\beta}{r} + \gamma r \]  

(15)
Figure 1: Some typical measured galactic rotation velocities
2 THE NON-TRIVIALITY OF 2D QUANTUM EINSTEIN GRAVITY

2D CLASSICAL \((-g)^{1/2} R^\alpha_\alpha\) IS A TOTAL DIVERGENCE (GAUSS-BONNET THEOREM) AND \(G_{\mu\nu} = 0\) ON EVERY CLASSICAL PATH.

BUT IN QUANTUM ORDERING MATTERS: \(A\partial_\mu B + B\partial_\mu A = \partial_\mu (AB) + [B, \partial_\mu A] \)

\[
R_{\lambda\mu\nu\kappa} = \frac{1}{2} \left( \partial_\kappa \partial_\mu g_{\lambda\nu} - \partial_\kappa \partial_\lambda g_{\mu\nu} - \partial_\nu \partial_\mu g_{\lambda\kappa} + \partial_\nu \partial_\lambda g_{\mu\kappa} \right) + g_{\eta\sigma} (\Gamma^\eta_{\nu\lambda} \Gamma^\sigma_{\mu\kappa} - \Gamma^\eta_{\nu\kappa} \Gamma^\sigma_{\mu\lambda}), \tag{16}
\]

\[
G_{\mu\nu}(1) = 0. \tag{17}
\]

\[
G_{00}(2) = \frac{1}{4} [\partial_0 h_{00}, \partial_1 h_{01}] + \frac{1}{4} [\partial_1 h_{11}, \partial_0 h_{01}] + \frac{1}{8} [\partial_0 h_{11}, \partial_0 h_{00}] + \frac{1}{8} [\partial_1 h_{00}, \partial_1 h_{11}] = G_{11}(2),
\]

\[
G_{01}(2) = \frac{1}{8} [\partial_0 h_{00}, \partial_1 h_{00}] + \frac{1}{8} [\partial_1 h_{11}, \partial_0 h_{11}] + \frac{1}{4} [\partial_1 h_{00}, \partial_1 h_{01}] + \frac{1}{4} [\partial_0 h_{11}, \partial_0 h_{01}], \tag{18}
\]

\[
\partial_\mu G^\mu_{\ 00}(2) = \frac{1}{4} [\nabla^2 h_{00}, \partial_1 h_{01}] - \frac{1}{4} [\nabla^2 h_{01}, \partial_1 h_{11}] - \frac{1}{4} [\partial_0 \partial_1 h_{01}, \partial_0 h] - \frac{1}{8} [\partial_0 \partial_1 (h_{00} + h_{11}), \partial_1 h] + \frac{1}{4} [\partial_1^2 h_{01}, \partial_1 h] + \frac{1}{8} [\nabla^2 h, \partial_0 h_{00}] + \frac{1}{8} [\partial_1^2 h_{11}, \partial_0 h] + \frac{1}{8} [\partial_0^2 h_{00}, \partial_0 h],
\]

\[
\partial_\mu G^\mu_{\ 01}(2) = \frac{1}{4} [\nabla^2 h_{11}, \partial_0 h_{01}] - \frac{1}{4} [\nabla^2 h_{01}, \partial_0 h_{00}] - \frac{1}{4} [\partial_0 \partial_1 h_{01}, \partial_1 h] - \frac{1}{8} [\partial_0 \partial_1 (h_{00} + h_{11}), \partial_0 h] + \frac{1}{4} [\partial_0^2 h_{01}, \partial_0 h] - \frac{1}{8} [\nabla^2 h, \partial_1 h_{11}] + \frac{1}{8} [\partial_0^2 h_{00}, \partial_1 h] + \frac{1}{8} [\partial_1^2 h_{11}, \partial_1 h], \tag{19}
\]

\[
\]
\[ I_{\text{GRAV}} = -\frac{1}{2\kappa^2} \int d^2 x (-g)^{1/2} R^\alpha_\alpha, \]

\[ I_M = -\frac{1}{2} \int d^2 x (-g)^{1/2} \left[ i\hbar \bar{\psi} \gamma^\mu(x) [\partial_\mu + \Gamma_\mu(x)] \psi - i\hbar \bar{\psi} [\partial_\mu + \Gamma_\mu(x)] \gamma^\mu(x) \psi \right], \quad (20) \]

\[ T_{\mu\nu}^M = \frac{i\hbar}{4} \bar{\psi} \gamma^\mu(x) [\partial^\nu + \Gamma^\nu(x)] \psi + \frac{i\hbar}{4} \bar{\psi} \gamma^\nu(x) [\partial^\mu + \Gamma^\mu(x)] \psi + \text{H. c.} \]

\[ -\frac{1}{2} g_{\mu\nu} \left[ i\hbar \bar{\psi} \gamma^\alpha(x) [\partial_\alpha + \Gamma_\alpha(x)] \psi - i\hbar \bar{\psi} [\partial_\alpha + \Gamma_\alpha(x)] \gamma^\alpha(x) \psi \right], \quad (21) \]

\[ \frac{1}{\kappa^2} G^{\mu\nu} + T_{\mu\nu}^M = 0. \quad (22) \]

\[ \nabla^2 h_{00} = 0, \quad \nabla^2 h_{01} = 0, \quad \nabla^2 h_{11} = 0, \quad h = -h_{00} + h_{11} = 0. \quad (23) \]
3 EXPLICIT DETAILS OF THE 2D CANCELLATION MECHANISM

\[
\begin{align*}
    u(k, \omega_k) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & u(-k, \omega_k) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\
    v(-k, -\omega_k) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & v(k, -\omega_k) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\end{align*}
\]

(24)

\[
\begin{align*}
    \psi^\dagger_\alpha(x, t) &= \int \frac{dk}{(2\pi)^{1/2}} \left[ b^\dagger(k) u^\dagger_\alpha(k, \omega_k) e^{-i(kx-\omega_k t)} + d(k) v^\dagger_\alpha(-k, -\omega_k) e^{i(kx-\omega_k t)} \right]. \\
    \psi_\alpha(x, t) &= \int \frac{dk}{(2\pi)^{1/2}} \left[ b(k) u_\alpha(k, \omega_k) e^{i(kx-\omega_k t)} + d^\dagger(k) v_\alpha(-k, -\omega_k) e^{-i(kx-\omega_k t)} \right].
\end{align*}
\]

(25)

\[
\begin{align*}
    \{ b(k), b^\dagger(k') \} &= \delta(k - k'), & \{ d(k), d^\dagger(k') \} &= \delta(k - k'), & \{ b(k), b(k') \} &= 0, \\
    \{ d(k), d(k') \} &= 0, & \{ b^\dagger(k), b^\dagger(k') \} &= 0, & \{ d^\dagger(k), d^\dagger(k') \} &= 0, \\
    \{ b(k), d(k') \} &= 0, & \{ b(k), d^\dagger(k') \} &= 0, & \{ d(k), b^\dagger(k') \} &= 0.
\end{align*}
\]

(26)
\[ T^{M}_{00} = \frac{\hbar}{4\pi} \int dk \int dk' \left[ (\omega_{k'} - \omega_k) d(k) b(k') v^\dagger(-k, -\omega_k) u(k', \omega_{k'}) e^{i(k+k')x} e^{-i(\omega_k + \omega_{k'})t} \right. \]

\[ - (\omega_{k'} - \omega_k) b^\dagger(k) d^\dagger(k') u^\dagger(k, \omega_k) v(-k', -\omega_{k'}) e^{-i(k+k')x} e^{i(\omega_k + \omega_{k'})t} \]

\[ + (\omega_{k'} + \omega_k) b^\dagger(k) b(k') u^\dagger(k, \omega_k) u(k', \omega_{k'}) e^{-i(k-k')x} e^{i(\omega_k - \omega_{k'})t} \]

\[ - (\omega_{k'} + \omega_k) d(k) d^\dagger(k') v^\dagger(-k, -\omega_k) v(-k', -\omega_{k'}) e^{i(k-k')x} e^{-i(\omega_k - \omega_{k'})t} \].

(27)

\[ H_M = \hbar \int_{-\infty}^{\infty} dk \omega_k \left[ b^\dagger(k) b(k) - d(k) d^\dagger(k) \right], \]

(28)

\[ \langle \Omega | T^M_{00} | \Omega \rangle = -\frac{\hbar}{2\pi} \int_{-\infty}^{\infty} dk \omega_k. \]

(29)

\[ \langle \Omega | T^M_{11} | \Omega \rangle = -\frac{\hbar}{2\pi} \int_{-\infty}^{\infty} dk \omega_k, \]

(30)
\[ h_{00}(x, t) = \kappa_2 \hbar^{1/2} \int \frac{dk}{(2\pi)^{1/2}(2\omega_k)^{1/2}} \left[ A(k)e^{i(kx-\omega_k t)} + C(k)e^{-i(kx-\omega_k t)} \right] = h_{11}(x, t), \]
\[ h_{01}(x, t) = \kappa_2 \hbar^{1/2} \int \frac{dk}{(2\pi)^{1/2}(2\omega_k)^{1/2}} \left[ B(k)e^{i(kx-\omega_k t)} + D(k)e^{-i(kx-\omega_k t)} \right], \]  
(31)

\[ G_{00}(2) = \frac{\kappa_2^2 \hbar}{16\pi} \int dk \int dk' \frac{(\omega_k k' + k\omega_{k'})}{(\omega_k \omega_{k'})^{1/2}} \]
\[ \times \left( [A(k), B(k')] e^{i(k+k')x} e^{-i(\omega_k + \omega_{k'})t} + [C(k), D(k')] e^{-i(k+k')x} e^{i(\omega_k + \omega_{k'})t} \right. \]
\[ \left. - [C(k), B(k')] e^{-i(k-k')x} e^{i(\omega_k - \omega_{k'})t} - [A(k), D(k')] e^{i(k-k')x} e^{-i(\omega_k - \omega_{k'})t} \right). \]  
(32)

\[ \langle \Omega | [C(k), B(k')] | \Omega \rangle = - \langle \Omega | B(k)C(k) | \Omega \rangle \delta(k-k') = - f_{BC}(k) \delta(k-k'), \]
\[ \langle \Omega | [A(k), D(k')] | \Omega \rangle = \langle \Omega | A(k)D(k) | \Omega \rangle \delta(k-k') = f_{AD}(k) \delta(k-k'), \]  
(33)

\[ \frac{1}{\kappa_2^2} \langle \Omega | G_{00}(2) | \Omega \rangle = \frac{\hbar}{8\pi} \int_{-\infty}^{\infty} dk k(f_{BC}(k) - f_{AD}(k)), \]
\[ \frac{1}{\kappa_2^2} \langle \Omega | G_{01}(2) | \Omega \rangle = \frac{\hbar}{8\pi} \int_{-\infty}^{\infty} dk \omega_k(f_{BC}(k) - f_{AD}(k)). \]  
(34)

\[ \frac{\hbar}{8\pi} \int_{-\infty}^{\infty} dk k(f_{BC}(k) - f_{AD}(k)) = \frac{\hbar}{2\pi} \int_{-\infty}^{\infty} dk \omega_k, \]
\[ \frac{\hbar}{8\pi} \int_{-\infty}^{\infty} dk \omega_k(f_{BC}(k) - f_{AD}(k)) = 0, \]  
(35)

\[ k(f_{BC}(k) - f_{AD}(k)) = 4\omega_k = 4|k|. \]  
(36)
\[
\frac{1}{\kappa^2_2} \int_{-\infty}^{\infty} dx G_{00}(2) = -\frac{\hbar}{4} \int_{-\infty}^{\infty} dk k \left( [C(k), B(k)] + [A(k), D(k)] \right) \\
= -\hbar \int_{-\infty}^{\infty} dk \omega_k \left[ b^\dagger(k) b(k) - d(k) d^\dagger(k) \right] 
\] (37)

\[
-\frac{\hbar}{4} \int_{-\infty}^{\infty} dk k \left( A^\dagger(k) B(k) - B(k) A^\dagger(k) + A(k) B^\dagger(k) - B^\dagger(k) A(k) \right) \\
= -\hbar \int_{-\infty}^{\infty} dk \omega_k \left[ b^\dagger(k) b(k) - d(k) d^\dagger(k) \right] .
\] (38)
THE GENERAL NATURE OF THE ZERO-POINT PROBLEM

\[ \langle \Omega | T_{M}^{00} | \Omega \rangle = -\frac{\hbar}{2\pi} \int_{-\infty}^{\infty} dk \omega_k, \quad \langle \Omega | T_{M}^{01} | \Omega \rangle = 0, \]  

\[ \langle \Omega | T_{M}^{\mu\nu} | \Omega \rangle = (\rho \text{M} + p \text{M}) U^{\mu} U^{\nu} + p \text{M} \eta^{\mu\nu} \]  

\[ \eta^{\mu\nu} \langle \Omega | T_{M}^{\mu\nu} | \Omega \rangle = 0, \quad p \text{M} - \rho \text{M} = 0 \]  

\[ E | \Omega \rangle = \int dx T_{00} | \Omega \rangle = \gamma E | \Omega \rangle, \quad P_{i} | \Omega \rangle = \int dx T_{0i} | \Omega \rangle \]  

\[ \epsilon(m) - \epsilon(m = 0) = (i/\hbar) \int d^{D}p/(2\pi)^{D}[\text{TrLn}(\gamma^{\mu} p_{\mu} - m + i\epsilon) - \text{TrLn}(\gamma^{\mu} p_{\mu} + i\epsilon)] = -\infty \]  

\[ T_{\text{UNIV}}^{\mu\nu} = T_{\text{GRAV}}^{\mu\nu} + T_{M}^{\mu\nu} = 0, \]  

\[ g_{\mu\nu} T_{\text{GRAV}}^{\mu\nu} = 0, \quad g_{\mu\nu} T_{M}^{\mu\nu} = 0. \]
5 MASS GENERATION AND THE COSMOLOGICAL CONSTANT PROBLEM

\[ I_{\text{UNIV}} = I_{\text{GRAV}} + I_{\text{M}} + \frac{g}{2} [\bar{\psi} \psi]^2, \]  
(41)

\[ T_{\text{M}}^{\mu\nu} = i\hbar \bar{\psi} \gamma^\mu \partial^\nu \psi - \eta^{\mu\nu} (g/2) [\bar{\psi} \psi]^2, \quad \eta_{\mu\nu} T_{\text{M}}^{\mu\nu} = i\hbar \bar{\psi} \gamma^\mu \partial_\mu \psi - g [\bar{\psi} \psi]^2 = 0. \]  
(42)

\[ i\hbar \gamma^\mu \partial_\mu \psi - m\psi = 0, \quad \langle S | T_{\text{MF}}^{\mu\nu} | S \rangle = \langle S | i\hbar \bar{\psi} \gamma^\mu \partial^\nu \psi | S \rangle - \frac{m^2}{2g} \eta^{\mu\nu}, \]

\[ \eta_{\mu\nu} \langle S | T_{\text{MF}}^{\mu\nu} | S \rangle = m \langle S | \bar{\psi} \psi | S \rangle - \frac{m^2}{g} = 0, \]  
(43)

\[ \langle S | i\hbar \bar{\psi} \gamma^0 \partial^0 \psi | S \rangle = -\frac{\hbar}{2\pi} \int_{-\infty}^{\infty} dk \omega_k, \quad \langle S | i\hbar \bar{\psi} \gamma^0 \partial^1 \psi | S \rangle = 0, \]

\[ \langle S | i\hbar \bar{\psi} \gamma^1 \partial^1 \psi | S \rangle = -\frac{\hbar}{2\pi} \int_{-\infty}^{\infty} dk \frac{k^2}{\omega_k}, \]  
(44)

\[ m \langle S | \bar{\psi} \psi | S \rangle = -\frac{\hbar}{2\pi} \int_{-K}^{K} dk \left( \frac{m^2}{\hbar^2 \omega_k} \right) = -\frac{m^2}{\pi \hbar} \ln \left( \frac{2\hbar K}{m} \right) = \frac{m^2}{g} = 2 \Lambda_{\text{MF}}, \]  
(45)

\[ m = 2\hbar K e^{-\pi h/g}. \]  
(46)

\[ \frac{1}{\kappa^2} G^{\mu\nu} + T_{\text{MF}}^{\mu\nu} = 0, \]  
(47)
\[
\rho_{MF} = -\frac{\hbar}{2\pi} \int_{-K}^{K} dk \omega_k = -\frac{\hbar}{2\pi} \left[ K^2 + \frac{m^2}{2\hbar^2} + \frac{m^2}{\hbar^2} \ln \left( \frac{2\hbar K}{m} \right) \right],
\]
\[
p_{MF} = -\frac{\hbar}{2\pi} \int_{-K}^{K} dk \frac{k^2}{\omega_k} = -\frac{\hbar}{2\pi} \left[ K^2 + \frac{m^2}{2\hbar^2} - \frac{m^2}{\hbar^2} \ln \left( \frac{2\hbar K}{m} \right) \right].
\]

(48)

\[
\langle S| T_{\mu\nu}^{MF} |S \rangle = (\rho_{MF} + p_{MF}) U^\mu U^\nu + p_{MF} \eta^{\mu\nu} - \eta^{\mu\nu} \Lambda_{MF},
\]
\[
\eta^{\mu\nu} \langle S| T_{\mu\nu}^{MF} |S \rangle = p_{MF} - \rho_{MF} - 2 \Lambda_{MF} = 0,
\]

(49)

\[
\Lambda_{MF} = \frac{m^2}{2\pi \hbar} \ln \left( \frac{2\hbar K}{m} \right).
\]

(50)

\[
\langle S| T_{\mu\nu}^{MF} |S \rangle = (\rho_{MF} + p_{MF}) \left[ U^\mu U^\nu + \frac{1}{2} \eta^{\mu\nu} \right],
\]

(51)

\[
\rho_{MF} + p_{MF} = -\frac{\hbar}{\pi} \left( K^2 + \frac{m^2}{2\hbar^2} \right),
\]

(52)

\[
\frac{1}{\kappa^2} \langle \Omega| G^{\mu\nu}(2)|\Omega \rangle = (\rho_{GRAV} + p_{GRAV}) \left[ U^\mu U^\nu + \frac{1}{2} \eta^{\mu\nu} \right].
\]

(53)

\[
\frac{\hbar}{8\pi} \int_{-K}^{K} dk k(f_{BC}(k) - f_{AD}(k)) = \frac{\hbar}{4\pi} \int_{-K}^{K} dk \left[ \omega_k + \frac{k^2}{\omega_k} \right] = \frac{\hbar}{2\pi} \left( K^2 + \frac{m^2}{2\hbar^2} \right).
\]

(54)

\[
k(f_{BC}(k) - f_{AD}(k)) = 4|k|, \quad k(f_{BC}(k) - f_{AD}(k)) = 4 \left[ (k^2 + m^2)^{1/2} - \frac{m^2}{2(k^2 + m^2)^{1/2}} \right].
\]

(55)
ALL YOU NEED TO KNOW ABOUT CONFORMAL GRAVITY

\[ g_{\mu\nu} \rightarrow e^{2\alpha(x)} g_{\mu\nu}, \quad C_{\lambda\mu\nu\kappa} \rightarrow e^{2\alpha(x)} C_{\lambda\mu\nu\kappa} \]  \hspace{1cm} (56)

\[ ds^2 \rightarrow e^{2\alpha(x)} ds^2, \quad ds^2 = 0 \rightarrow ds^2 = 0 \]  \hspace{1cm} (57)

\[ C_{\lambda\mu\nu\kappa} = R_{\lambda\mu\nu\kappa} + \frac{1}{6} R^\alpha_{\lambda} \left[ g_{\lambda\nu} g_{\mu\kappa} - g_{\lambda\kappa} g_{\mu\nu} \right] - \frac{1}{2} \left[ g_{\lambda\nu} R_{\mu\kappa} - g_{\lambda\kappa} R_{\mu\nu} - g_{\mu\nu} R_{\lambda\kappa} + g_{\mu\kappa} R_{\lambda\nu} \right] \]  \hspace{1cm} (58)

\[ I_W = -\alpha g \int d^4 x (-g)^{1/2} C_{\lambda\mu\nu\kappa} C^{\lambda\mu\nu\kappa} = -2\alpha g \int d^4 x (-g)^{1/2} \left[ R^{\mu\nu} R_{\mu\nu} - \frac{1}{3} (R^\alpha_{\alpha})^2 \right] \]  \hspace{1cm} (59)

\[ -4\alpha g W^{\mu\nu} + T^{\mu\nu} = 0 \]  \hspace{1cm} (60)

\[ W^{\mu\nu} = \frac{1}{2} g^{\mu\nu} (R^\alpha_{\alpha})_{;\beta} + R^{\mu\nu;\beta} - R^{\mu\beta;\nu} - R^{\nu\beta;\mu} - 2R^{\mu\beta} R_{\nu\beta} + \frac{1}{2} g^{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} - \frac{2}{3} g^{\mu\nu} (R^\alpha_{\alpha})_{;\beta} + \frac{2}{3} (R^\alpha_{\alpha})^{;\mu\nu} + \frac{2}{3} R^\alpha_{\alpha} R^{\mu\nu} - \frac{1}{6} g^{\mu\nu} (R^\alpha_{\alpha})^2 = 2C^{\mu\lambda\nu\kappa}_{;\lambda;\kappa} - C^{\mu\lambda\nu\kappa} R_{\lambda\kappa} \]  \hspace{1cm} (61)

\[ g_{\mu\nu} W^{\mu\nu} = 0, \quad g_{\mu\nu} T^{\mu\nu} = 0, \quad T^{\mu\nu}_{\text{UNIV}} = T^{\mu\nu}_{\text{GRAV}} + T^{\mu\nu}_{\text{MAT}} = 0. \]  \hspace{1cm} (62)

Outside static source : \[ -g_{00} = \frac{1}{g_{rr}} = 1 - \frac{2\beta}{r} + \gamma r \]  \hspace{1cm} (63)
Figure 2: Some typical measured galactic rotation velocities
ALL YOU NEED TO KNOW ABOUT STANDARD COSMOLOGY

\[ ds^2 = -c^2dt^2 + R^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2 \right] \]  \hfill (64)

\[ H = \frac{\dot{R}}{R}, \quad q = -\frac{R\ddot{R}}{\dot{R}^2} \]  \hfill (65)

\[ T_{\mu\nu} = (\rho_M + p_M)U_\mu U_\nu + p_M g_{\mu\nu} - \Lambda g_{\mu\nu}, \quad \rho_M = \frac{A}{R^n} \]  \hfill (66)

\[ \dot{R}^2 + kc^2 = \dot{R}^2 [\Omega_M + \Omega_\Lambda] \]  \hfill (67)

\[ q = -\frac{R\ddot{R}}{\dot{R}^2} = \left(\frac{n}{2} - 1\right) \Omega_M - \Omega_\Lambda \]  \hfill (68)

\[ \Omega_M = \frac{8\pi G \rho_M}{3c^2H^2}, \quad \Omega_\Lambda = \frac{8\pi G \Lambda}{3cH^2}, \]  \hfill (69)

\[ \frac{\Omega_\Lambda}{\Omega_M} = \frac{c \Lambda}{\rho_M} = \frac{T_V^4}{T^4} \]  \hfill (70)
ALL YOU NEED TO KNOW ABOUT CONFORMAL COSMOLOGY

\[ I_M = -\int d^4x (-g)^{1/2} \left[ \frac{1}{2} S^\mu ;_\nu S ;_\mu - \frac{1}{12} S^2 R^\mu \_\mu + \lambda S^4 + i \bar{\psi} \gamma^\mu [\partial_\mu + \Gamma_\mu] \psi - h S \bar{\psi} \psi \right] \]

\[ T_{\mu \nu} = (\rho_M + p_M) U_\mu U_\nu + p_M g_{\mu \nu} - \frac{1}{6} S_0^2 \left( R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R^\alpha _\alpha \right) - g_{\mu \nu} \lambda S^4_0 \]

\[ C_{\lambda \mu \nu \kappa} = 0, \quad T_{\mu \nu} = 0 \]

\[ \frac{1}{6} S_0^2 \left( R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R^\alpha _\alpha \right) = (\rho_M + p_M) U_\mu U_\nu + p_M g_{\mu \nu} - g_{\mu \nu} \lambda S^4_0 \]

\[ G_{\text{eff}} = -\frac{3}{4 \pi S_0^2} \]

\[ \dot{R}^2 + kc^2 = \dot{R}^2 \left[ \bar{\Omega}_M + \bar{\Omega}_\Lambda \right], \quad q = -\frac{R \ddot{R}}{R^2} = \left( \frac{n}{2} - 1 \right) \bar{\Omega}_M - \bar{\Omega}_\Lambda \]

\[ \bar{\Omega}_M = \frac{8 \pi G_{\text{eff}} \rho_M}{3 c^2 H^2}, \quad \bar{\Omega}_\Lambda = \frac{8 \pi G_{\text{eff}} \Lambda}{3 c^2 H^2}, \quad \frac{\bar{\Omega}_\Lambda}{\bar{\Omega}_M} = \frac{c \Lambda}{\rho_M} = -\frac{T^4_V}{T^4} \]

\[ \bar{\Omega}_\Lambda = \left( 1 - \frac{T^2}{T^2_{\text{max}}} \right)^{-1} \left( 1 + \frac{T^2 T^2_{\text{max}}}{T^4_V} \right)^{-1}, \quad \bar{\Omega}_M = -\frac{T^4}{T^4_V} \bar{\Omega}_\Lambda \]

\[ 0 \leq \bar{\Omega}_\Lambda \leq 1, \quad -1 \leq q \leq 0 \]

\[ d_L = -\frac{c}{H_0} \frac{(1 + z)^2}{q_0} \left( 1 - \left[ 1 + q_0 - \frac{q_0}{(1 + z^2)} \right]^{1/2} \right) \]
$(\Omega_m, \Omega_\Lambda) =$

- $(0, 1)$
- $(0.5, 0.5)$
- $(1, 0)$
- $(1.5, -0.5)$
- $(2, 0)$

$\Lambda = 0$

Flat

Supernova Cosmology Project

Calan/Tololo (Hamuy et al, A.J. 1996)

$z$

$M_B$

$z$

$M_B$

$z$

$M_B$

$z$

$M_B$
No Big Bang

$\Omega_{\Lambda}$

$\Omega_{M}$

expands forever

recollapses eventually

Flat
$\Lambda = 0$

Universe

Closed

Open

25
Figure 5: The $q_0 = -0.37$ conformal gravity fit (upper curve) and the $\Omega_M(t_0) = 0.3$, $\Omega_\Lambda(t_0) = 0.7$ standard model fit (lower curve) to the $z < 1$ supernovae Hubble plot data.
Figure 6: Hubble plot expectations for $q_0 = -0.37$ (highest curve) and $q_0 = 0$ (middle curve) conformal gravity and for $\Omega_M(t_0) = 0.3$, $\Omega_\Lambda(t_0) = 0.7$ standard gravity (lowest curve).
\[ I_W = -\alpha_g \int d^4x (-g)^{1/2} C_{\lambda\mu\nu\kappa} C^{\lambda\mu\nu\kappa} \]
\[ = -\alpha_g \int d^4x (-g)^{1/2} \left[ R_{\lambda\mu\nu\kappa} R^{\lambda\mu\nu\kappa} - 2 R_{\mu\kappa} R^{\mu\kappa} + \frac{1}{3} (R^\alpha_\alpha)^2 \right]. \quad (81) \]

\[ C_{\lambda\mu\nu\kappa} = R_{\lambda\mu\nu\kappa} - \frac{1}{2} (g_{\lambda\nu} R_{\mu\kappa} - g_{\lambda\kappa} R_{\mu\nu} - g_{\mu\nu} R_{\lambda\kappa} + g_{\mu\kappa} R_{\lambda\nu}) + \frac{1}{6} R^\alpha_\alpha (g_{\lambda\nu} g_{\mu\kappa} - g_{\lambda\kappa} g_{\mu\nu}). \quad (82) \]

\[ \frac{1}{(-g)^{1/2}} \frac{\delta I_W}{\delta g_{\mu\nu}} = -2\alpha_g W^{\mu\nu} \quad (83) \]

\[ L_L = (-g)^{1/2} \left[ R_{\lambda\mu\nu\kappa} R^{\lambda\mu\nu\kappa} - 4 R_{\mu\kappa} R^{\mu\kappa} + (R^\alpha_\alpha)^2 \right] \quad (84) \]

\[ I_W = -2\alpha_g \int d^4x (-g)^{1/2} \left[ R_{\mu\kappa} R^{\mu\kappa} - \frac{1}{3} (R^\alpha_\alpha)^2 \right], \quad (85) \]

\[ 4\alpha_g W^{\mu\nu} = 4\alpha_g \left[ W^{\mu\nu}_{(2)} - \frac{1}{3} W^{\mu\nu}_{(1)} \right] = T^{\mu\nu}_M, \quad (86) \]

\[ W^{\mu\nu}_{(1)} = 2g^{\mu\nu} (R^\alpha_\alpha)^{;\beta} ;\beta - 2 (R^\alpha_\alpha)^{;\mu;\nu} - 2 R^\alpha_\alpha R^{\mu\nu} + \frac{1}{2} g^{\mu\nu} (R^\alpha_\alpha)^2, \quad (87) \]

\[ W^{\mu\nu}_{(2)} = \frac{1}{2} g^{\mu\nu} (R^\alpha_\alpha)^{;\beta} ;\beta + R^{\mu\nu;\beta} ;\beta - R^{\mu;\beta;\nu} ;\beta - R^{\nu;\beta;\mu} ;\beta - 2 R^{\mu\beta} R^{\nu}_\beta + \frac{1}{2} g^{\mu\nu} R^{\alpha\beta} R^\alpha_\beta. \quad (88) \]
\[ W^{\mu\nu}(1) = \frac{1}{2} \Pi^{\mu\rho} \Pi^{\nu\sigma} K_{\rho\sigma} - \frac{1}{6} \Pi^{\mu\nu} \Pi^{\rho\sigma} K_{\rho\sigma}, \quad (89) \]

\[ K^{\mu\nu} = h^{\mu\nu} - \frac{1}{4} \eta^{\mu\nu} h^{\alpha\alpha}, \quad \Pi^{\mu\nu} = \eta^{\mu\nu} \partial^{\alpha} \partial_{\alpha} - \partial^{\mu} \partial^{\nu}. \quad (90) \]

\[ W^{\mu\nu}(1) = \frac{1}{2} (\partial^{\alpha} \partial^{\alpha}) K^{\mu\nu}, \quad (91) \]

\[ I_W = -\frac{\alpha_g}{2} \int d^4 x \partial^{\alpha} K_{\mu\nu} \partial^{\beta} K^{\mu\nu}, \quad (92) \]

\[ K_{\mu\nu} = A_{\mu\nu} e^{ik \cdot x} + B_{\mu\nu} (n \cdot x) e^{ik \cdot x} \quad (93) \]

\[ K_{\mu\nu}(x) = \frac{\hbar^{1/2}}{2(-\alpha_g)^{1/2}} \sum_i \int \frac{d^3 k}{(2\pi)^3/2(\omega_k)^{3/2}} \left[ A^{(i)}(\bar{k}) \epsilon^{(i)}_{\mu\nu}(\bar{k}) e^{ik \cdot x} + i\omega_k B^{(i)}(\bar{k}) \epsilon^{(i)}_{\mu\nu}(\bar{k})(n \cdot x) e^{ik \cdot x} \right. \]
\[ + \left. \hat{A}^{(i)}(\bar{k}) \epsilon^{(i)}_{\mu\nu}(\bar{k}) e^{-ik \cdot x} - i\omega_k \hat{B}^{(i)}(\bar{k}) \epsilon^{(i)}_{\mu\nu}(\bar{k})(n \cdot x) e^{-ik \cdot x} \right], \quad (94) \]
\[-4\alpha_g \langle \Omega | W_{\mu\nu}(2) | \Omega \rangle = \hbar \int \frac{d^3k}{(2\pi)^3\omega_k} \langle \Omega | X_{\mu\nu}(k) | \Omega \rangle, \tag{95} \]

\[X_{\mu\nu}(k) = \sum_i \left[ \left\{ \frac{1}{2}k_\mu k_\nu(n \cdot n) - (k_\mu n_\nu + k_\nu n_\mu)(k \cdot n) + \frac{1}{2}\eta_{\mu\nu}(k \cdot n)^2 \right\} \left( \hat{B}^{(i)} B^{(i)} + B^{(i)} \hat{B}^{(i)} \right) \right. \]
\[\left. - \frac{k_\mu k_\nu}{\omega_k}(k \cdot n) \left[ \hat{A}^{(i)} B^{(i)} + A^{(i)} \hat{B}^{(i)} \right] + ik_\mu k_\nu(k \cdot n)(\hat{B}^{(i)} B^{(i)} - B^{(i)} \hat{B}^{(i)})(n \cdot x) \right]. \tag{96} \]

\[\hat{B}^{(1)}(\bar{k}) B^{(1)}(\bar{k}) - B^{(1)}(\bar{k}) \hat{B}^{(1)}(\bar{k}) = 0, \quad \hat{B}^{(2)}(\bar{k}) B^{(2)}(\bar{k}) - B^{(2)}(\bar{k}) \hat{B}^{(2)}(\bar{k}) = 0. \tag{97} \]

\[X_{\mu\nu}(k) = \sum_i \left[ k_\mu k_\nu(n \cdot n) - 2(k_\mu n_\nu + k_\nu n_\mu)(k \cdot n) + \eta_{\mu\nu}(k \cdot n)^2 \right] \hat{B}^{(i)}(\bar{k}) B^{(i)}(\bar{k}) \]
\[\left. - \frac{k_\mu k_\nu}{\omega_k}(k \cdot n) \left[ \hat{A}^{(i)}(\bar{k}) B^{(i)}(\bar{k}) + A^{(i)}(\bar{k}) \hat{B}^{(i)}(\bar{k}) \right] \right]. \tag{98} \]

\[-4\alpha_g \int d^3 x W_{00}(2) \]
\[= \sum_i \int d^3 k \hbar \omega_k \left[ \hat{A}^{(i)}(\bar{k}) B^{(i)}(\bar{k}) + A^{(i)}(\bar{k}) \hat{B}^{(i)}(\bar{k}) + 2\hat{B}^{(i)}(\bar{k}) B^{(i)}(\bar{k}) \right], \tag{99} \]
GHOST PROBLEM OF SECOND- PLUS FOURTH-ORDER THEORIES

\[ I_S = -\frac{1}{2} \int d^4 x \left[ \partial_\mu \partial_\nu \phi \partial^\mu \partial^\nu \phi + (M_1^2 + M_2^2) \partial_\mu \phi \partial^\mu \phi + M_1^2 M_2^2 \phi^2 \right], \quad (100) \]

\[ (-\partial_t^2 + \nabla^2 - M_1^2)(-\partial_t^2 + \nabla^2 - M_2^2)\phi(x) = 0. \quad (101) \]

\[ D^{(4)}(x, M_1, M_2) = \int \frac{d^4 x}{(2\pi)^4} \frac{e^{ik \cdot x}}{(k^2 + M_1^2)(k^2 + M_2^2)} = \int \frac{d^4 x}{(2\pi)^4} \frac{e^{ik \cdot x}}{(M_2^2 - M_1^2)} \left( \frac{1}{k^2 + M_1^2} - \frac{1}{k^2 + M_2^2} \right). \quad (102) \]

\[ D^{(4)}(\bar{x}, \bar{x}', t, M_1, M_2) = \frac{1}{(M_1^2 - M_2^2)} \int \frac{d^3 k}{(2\pi)^3} \frac{e^{i k \cdot (\bar{x} - \bar{x}')}}{2\omega_k^1} \left[ \theta(t)e^{-i\omega_k^1 t} + \theta(-t)e^{i\omega_k^1 t} \right] \]

\[ - \frac{1}{(M_1^2 - M_2^2)} \int \frac{d^3 k}{(2\pi)^3} \frac{e^{i k \cdot (\bar{x} - \bar{x}')}}{2\omega_k^2} \left[ \theta(t)e^{-i\omega_k^2 t} + \theta(-t)e^{i\omega_k^2 t} \right]. \quad (103) \]

\[ \sum_n |n\rangle\langle n| - \sum_m |m\rangle\langle m| = 1, \quad (104) \]

\[ D^{(4)}(\bar{x}, \bar{x}', E, M_1, M_2) = \frac{1}{(M_2^2 - M_1^2)} \left( D^{(2)}(\bar{x}, \bar{x}', E, M_1) - D^{(2)}(\bar{x}, \bar{x}', E, M_2) \right), \quad (105) \]

\[ H|R\rangle = E|R\rangle, \quad \langle L|H = \langle L|E, \quad (106) \]
NON-HERMITICITY AND UNITARITY

\[ H = H^\dagger \quad H|R\rangle = E|R\rangle, \quad \langle R|H = \langle R|E \]

\[ \langle R(t)|R(t)\rangle = \langle R(t = 0)e^{iHt}|e^{-iHt}R(t = 0)\rangle = \langle R(t = 0)|R(t = 0)\rangle \]

\[ H \neq H^\dagger \quad H|R\rangle = E|R\rangle, \quad \langle R|H^\dagger = \langle R|E \]

\[ \langle R(t)|R(t)\rangle = \langle R(t = 0)e^{iH^\dagger t}|e^{-iHt}R(t = 0)\rangle \neq \langle R(t = 0)|R(t = 0)\rangle \]

\[ H \neq H^\dagger \quad H|R\rangle = E|R\rangle, \quad \langle L|H = \langle L|E \]

\[ \langle L(t)|R(t)\rangle = \langle L(t = 0)e^{iHt}|e^{-iHt}R(t = 0)\rangle = \langle L(t = 0)|R(t = 0)\rangle \]

\[ \langle L| = \langle R|S \quad \langle L|R\rangle = \langle R|S|R\rangle \]

\[ [C, H] = 0, \quad C^2 = 1, \quad S = PC = e^{-Q}, \quad \tilde{H} = e^{-Q/2}He^{Q/2} = \tilde{H}^\dagger \]
\[\pi^\mu = \frac{\partial L}{\partial \phi, \mu} - \partial_\lambda \left( \frac{\partial L}{\partial \phi, \mu, \lambda} \right) = -(M_1^2 + M_2^2) \partial^\mu \phi + \partial_\lambda \partial^\mu \partial^\lambda \phi, \quad \pi^{\mu \lambda} = \frac{\partial L}{\partial \phi, \mu, \lambda} = -\partial^\mu \partial^\lambda \phi, \quad (107)\]

\[T_{\mu \nu}^{\text{OST}}(M_1, M_2) = \pi_{\mu \nu} + \pi_{\mu \lambda} \phi_{, \nu}^\lambda - \eta_{\mu \nu} L = \pi_{\mu \nu} + \pi_{\mu \lambda} \pi_{\nu}^\lambda + \frac{1}{2} \eta_{\mu \nu} \left[ \pi_{\lambda \kappa} \pi^{\lambda \kappa} + (M_1^2 + M_2^2) \partial_\lambda \phi \partial_\nu \phi + M_1^2 M_2^2 \phi^2 \right], \quad (108)\]

\[T_{00}^{\text{OST}}(M_1, M_2) = \pi_0 \dot{\phi} + \frac{1}{2} \left[ \pi_{00}^2 + (M_1^2 + M_2^2)(\dot{\phi}^2 - \partial_i \phi \partial^i \phi) - M_1^2 M_2^2 \phi^2 - \pi_{ij} \pi^{ij} \right]. \quad (108)\]

\[\phi(x) = \hbar^{1/2} \int \frac{d^3 k}{(2\pi)^{3/2}(2\omega_k^{1/2})^{1/2}} [a_{1, k} e^{i(k \cdot \bar{x} - \omega_k^1 t)} + \hat{a}_{1, k} e^{-i(k \cdot \bar{x} - \omega_k^1 t)}] + \hbar^{1/2} \int \frac{d^3 k}{(2\pi)^{3/2}(2\omega_k^{2})^{1/2}} [i a_{2, k} e^{i(k \cdot \bar{x} - \omega_k^2 t)} + i \hat{a}_{2, k} e^{-i(k \cdot \bar{x} - \omega_k^2 t)}] \quad (109)\]

\[\begin{align*}
[a_{1, \bar{k}}, \hat{a}_{1, \bar{k}'}] &= \frac{1}{(M_1^2 - M_2^2)} \delta^3(\bar{k} - \bar{k'}), \\
[a_{2, \bar{k}}, \hat{a}_{2, \bar{k}'}] &= \frac{1}{(M_1^2 - M_2^2)} \delta^3(\bar{k} - \bar{k'}), \\
[a_{1, \bar{k}}, a_{2, \bar{k}'}] &= 0, \\
[\hat{a}_{1, \bar{k}}, \hat{a}_{2, \bar{k}'}] &= 0, \\
[a_{1, \bar{k}}, \hat{a}_{2, \bar{k}'}] &= 0, \\
[\hat{a}_{1, \bar{k}}, a_{2, \bar{k}'}] &= 0, \\
[\hat{a}_{1, \bar{k}}, \hat{a}_{2, \bar{k}'}] &= 0, \quad (110)\end{align*}\]

\[H^{\text{OST}}(M_1, M_2) = \hbar \int d^3 k \left[ (M_1^2 - M_2^2) [\omega_k^1 a_{1, \bar{k}} a_{1, \bar{k}} + \omega_k^2 a_{2, \bar{k}} a_{2, \bar{k}}] + \frac{1}{2} (\omega_k^1 + \omega_k^2) \delta^3(0) \right], \quad (111)\]
8  THE GHOST PROBLEM OF PURE FOURTH-ORDER THEORIES

\[
\psi_1 \rightarrow e^{i(k \cdot x - \omega_k t)} \left( 1 - \frac{i M_1^2 t}{2 \omega_k} \right), \quad \psi_2 \rightarrow e^{i(k \cdot x - \omega_k t)} \left( 1 - \frac{i M_2^2 t}{2 \omega_k} \right).
\]

(112)

\[
\psi_- = \frac{2 i \omega_k}{(M_1^2 - M_2^2)} (\psi_1 - \psi_2) \rightarrow e^{i(k \cdot x - \omega_k t)} t.
\]

(113)

\[
M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.
\]

(114)

\[
\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (0 \ 1) = (0 \ 1) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.
\]

(115)

\[
a_{\bar{k}} = \mu_{\bar{k}}(a_{1,\bar{k}} + i a_{2,\bar{k}}) + \frac{b_{\bar{k}}}{2}, \quad \hat{a}_{\bar{k}} = \mu_{\bar{k}}(\hat{a}_{1,\bar{k}} + i \hat{a}_{2,\bar{k}}) + \frac{\hat{b}_{\bar{k}}}{2},
\]

\[
b_{\bar{k}} = \lambda_{\bar{k}}(a_{1,\bar{k}} - i a_{2,\bar{k}}), \quad \hat{b}_{\bar{k}} = \lambda_{\bar{k}}(\hat{a}_{1,\bar{k}} - i \hat{a}_{2,\bar{k}}),
\]

\[
\mu_{\bar{k}} = \frac{\gamma_{\bar{k}}(\omega_{1,\bar{k}} + \omega_{2,\bar{k}})}{\sqrt{2}}, \quad \lambda_{\bar{k}} = \frac{\gamma_{\bar{k}}(\omega_{1,\bar{k}} - \omega_{2,\bar{k}})}{\sqrt{2}},
\]

(116)
\[ H^{\text{OST}}(M_1 = 0, M_2 = 0) = \hbar \int d^3 k \omega_k [\hat{a}_k b_k + a_k \hat{b}_k + 2\hat{b}_k b_k] , \]
\[ = \hbar \int d^3 k \omega_k [\hat{a}_k b_k + \hat{b}_k a_k + 2\hat{b}_k b_k + \delta^3(0)] , \] (117)

\[ \phi(x) = \int \frac{d^3 k \hbar^{1/2}}{(2\pi)^{3/2}(\omega_k)^{3/2}} [e^{i(\vec{k} \cdot \vec{x} - \omega_k t)} [a_k + i\omega_k (n \cdot x) b_k] + e^{-i(\vec{k} \cdot \vec{x} - \omega_k t)} [\hat{a}_k - i\omega_k (n \cdot x) \hat{b}_k] , \]
\[ \pi^0 = \partial_\lambda \partial^0 \partial^\lambda \phi, \quad \pi^{00} = -\partial^0 \partial^0 \phi , \] (118)

\[ [a_k, \hat{b}_k'] = [b_k, \hat{a}_k'] = \delta^3(\vec{k} - \vec{k}'), \]
\[ [a_k, \hat{a}_k'] = 0, \quad [b_k, \hat{b}_k'] = 0, \quad [a_k, b_k'] = 0, \quad [\hat{a}_k, b_k'] = 0 , \] (119)

\[ [A^{(i)}(\vec{k}), \hat{B}^{(j)}(\vec{k}')] = [B^{(i)}(\vec{k}), \hat{A}^{(j)}(\vec{k}')] = \delta_{i,j} \delta^3(\vec{k} - \vec{k}'), \]
\[ [A^{(i)}(\vec{k}), \hat{A}^{(j)}(\vec{k}')] = 0, \quad [B^{(i)}(\vec{k}), \hat{B}^{(j)}(\vec{k}')] = 0 , \]
\[ [A^{(i)}(\vec{k}), B^{(j)}(\vec{k}')] = 0, \quad [\hat{A}^{(i)}(\vec{k}), \hat{B}^{(j)}(\vec{k}')] = 0, \quad [\hat{A}^{(i)}(\vec{k}), B^{(j)}(\vec{k}')] = 0 \] (120)

\[ -4\alpha_g \int d^3 x W_{00}(2) \]
\[ = \sum_i \hbar \int d^3 k \omega_k \left[ \hat{A}^{(i)}(\vec{k}) B^{(i)}(\vec{k}) + \hat{B}^{(i)}(\vec{k}) A^{(i)}(\vec{k}) + 2\hat{B}^{(i)}(\vec{k}) B^{(i)}(\vec{k}) + \delta^3(0) \right] . \] (121)
9 CONCLUSIONS AND COMMENTS


\[ \tilde{\Gamma}_{\bar{\psi}\psi}(p, p, 0) = (-\frac{p^2}{M^2})^{-1/2}, \]

\[ \epsilon(m) = (i/\hbar) \int d^4p/(2\pi)^4 \text{Tr} \ln(\gamma^\mu p_\mu - m + i\epsilon) \]

\[ \epsilon(m) = (i/\hbar) \int d^4p/(2\pi)^4 \text{Tr} \ln(\gamma^\mu p_\mu - m(-\frac{p^2}{M^2})^{-1/2} + i\epsilon) \]

\[ M^4 = \hbar^4 K^4 \exp(8\pi^2 \hbar^3 / M^2 g) \]

\[ \epsilon(m) + \frac{m^2}{2g} = -\frac{\hbar K^4}{4\pi^2} + \frac{m^2 M^2}{16\pi^2 \hbar^3} \left[ \ln \left( \frac{m^2}{M^2} \right) - 1 \right], \]

\[ \epsilon(M) + \frac{M^2}{2g} = -\frac{\hbar K^4}{4\pi^2} - \frac{M^4}{16\pi^2 \hbar^3} \]  

(122)
DOING PHYSICS WITH NON-HERMITIAN HAMILTONIANS
AND NON-DIAGONALIZABLE HAMILTONIANS

Philip D. Mannheim
University of Connecticut

Presentation at Oxford University
June 2010
THINGS WE TAKE FOR GRANTED IN QUANTUM MECHANICS…..

(1) The Hamiltonian must be Hermitian.

(2) The momentum operator must be Hermitian.

(3) Self-adjoint is the same as Hermitian.

(4) The classical limit of a quantum theory must be based on real numbers.

(5) In the \([x, p] = i\hbar\) commutator the momentum operator can always be represented by \(p = -i\hbar \frac{\partial}{\partial x}\).

(6) States such as energy eigenstates must form a complete set.

(7) To be complete states must be normalizable.

(8) The scalar product must be given as \(\langle m|n \rangle = \delta_{m,n}\).

(9) The completeness relation must be given by \(\sum |n\rangle\langle n| = 1\).

(10) Theories in which \(\langle n|n \rangle\) is negative are unphysical and cannot be formulated in Hilbert space.

(11) The Hamiltonian must be diagonalizable.

…..AIN’T NECESSARILY SO

AND FOR THEORIES BASED ON FOURTH-ORDER DERIVATIVES....
ALL THESE THINGS ARE NECESSARILY NOT SO, ......

AND CAN ENABLE FOURTH-ORDER DERIVATIVE CONFORMAL GRAVITY TO BE A CONSISTENT THEORY OF QUANTUM GRAVITY IN FOUR SPACETIME DIMENSIONS
THE QUANTUM GRAVITY UNITARITY PROBLEM

In four spacetime dimensions invariance under

\[ g_{\mu\nu}(x) \rightarrow e^{2\alpha(x)} g_{\mu\nu}(x) \]  \hspace{1cm} (123)

leads to a unique gravitational action

\[ I_W = -\alpha_g \int d^4x (\pm g)^{1/2} C_{\lambda\mu\nu\kappa} C^{\lambda\mu\nu\kappa} = -2\alpha_g \int d^4x (\pm g)^{1/2} \left[ R_{\mu\nu} R_{\mu\nu} - \frac{1}{3}(R^n)\right] \]  \hspace{1cm} (124)

where \( \alpha_g \) is dimensionless and

\[ C_{\lambda\mu\nu\kappa} = R_{\lambda\mu\nu\kappa} + \frac{1}{6} R^n_{\alpha} \left[ g_{\lambda\nu} g_{\mu\kappa} - g_{\lambda\kappa} g_{\mu\nu} \right] - \frac{1}{2} \left[ g_{\lambda\nu} R_{\mu\kappa} - g_{\lambda\kappa} R_{\mu\nu} - g_{\mu\nu} R_{\lambda\kappa} + g_{\mu\kappa} R_{\lambda\nu} \right] \] \hspace{1cm} (125)

is the conformal Weyl tensor. The associated gravitational equations of motion are the fourth-order derivative:

\[ 4\alpha_g [2C^{\mu\nu\kappa}_{\lambda\kappa ;\lambda;\kappa} - C^{\mu\nu\kappa}_{\lambda\mu\nu\kappa} R_{\lambda\kappa} ] = T^{\mu\nu} \] \hspace{1cm} (126)

Conformal gravity is thus a power-counting renormalizable theory of gravity since \( \alpha_g \) is dimensionless. Moreover, conformal gravity controls the cosmological constant. Specifically, in a Robertson-Walker cosmology we have \( C^{\mu\nu\kappa}_{\mu\nu\kappa} = 0 \), to yield

\[ T^{\mu\nu} = 0, \] \hspace{1cm} (127)

so unlike the double-well Higgs potential, conformal gravity knows where the zero of energy is. However, since the field equations are fourth-order derivative equations, the theory is thought to have negative norm ghost states and not be unitary.
To illustrate the issues involved, consider the typical second- plus fourth-order derivative theory:

\[
I = \frac{1}{2} \int d^4x \left[ \partial_\mu \partial_\nu \phi \partial^\mu \partial^\nu \phi - M^2 \partial_\mu \phi \partial^\mu \phi \right]
\]  
(128)

\[
(\partial_t^2 - \nabla^2)(\partial_t^2 - \nabla^2 + M^2)\phi(\vec{x}, t) = 0
\]  
(129)

\[
D^{(4)}(k^2 = -k_0^2 + \vec{k}^2) = \frac{1}{k^2(k^2 + M^2)} = \frac{1}{M^2} \left( \frac{1}{k^2} - \frac{1}{k^2 + M^2} \right).
\]  
(130)

Does the relative minus sign in propagator mean ghost states with negative norm and loss of unitarity, since anticipate that one can write the propagator as

\[
D(\vec{x}, \vec{x}', E) = \sum \frac{\psi_n(\vec{x})\psi_n^*(\vec{x}')}{E - E_n} - \sum \frac{\psi_m(\vec{x})\psi_m^*(\vec{x}')}{E - E_m},
\]  
(131)

and the completeness relation as

\[
\sum |n\rangle\langle n| - \sum |m\rangle\langle m| = 1.
\]  
(132)
PAIS-UHLENBECK OSCILLATOR

\[ \phi(\bar{x}, t) \sim z(t)e^{i\vec{k} \cdot \bar{x}}, \quad \omega_1 = (\vec{k}^2 + M^2)^{1/2}, \quad \omega_2 = |\bar{k}| \] (133)

\[ \frac{d^4 z}{dt^4} + (\omega_1^2 + \omega_2^2)\frac{d^2 z}{dt^2} + \omega_1 \omega_2 z = 0 \] (134)

\[ I_{PU} = \frac{\gamma}{2} \int dt [\ddot{z}^2 - (\omega_1^2 + \omega_2^2)\dot{z}^2 + \omega_1^2 \omega_2^2 z^2] \] (135)

\[ H_{PU} = \frac{p_x^2}{2\gamma} + p_z x + \frac{\gamma}{2} (\omega_1^2 + \omega_2^2) x^2 - \frac{\gamma}{2} \omega_1^2 \omega_2^2 z^2, \quad x = \dot{z} \] (136)

\[ [x, p_x] = i, \quad [z, p_z] = i \] (137)

\[ z = a_1 + a_1^\dagger + a_2 + a_2^\dagger, \]
\[ p_z = i\gamma \omega_1^2 \omega_2 (a_1 - a_1^\dagger) + i\gamma \omega_1 \omega_2 (a_2 - a_2^\dagger), \]
\[ x = -i\omega_1 (a_1 - a_1^\dagger) - i\omega_2 (a_2 - a_2^\dagger), \]
\[ p_x = -\gamma \omega_1^2 (a_1 + a_1^\dagger) - \gamma \omega_2^2 (a_2 + a_2^\dagger) \] (138)
\[ H_{PU} = 2\gamma(\omega_1^2 - \omega_2^2)(\omega_1^2 a_1^\dagger a_1 - \omega_2^2 a_2^\dagger a_2) + (\omega_1 + \omega_2)/2 \quad (139) \]

\[ [a_1, a_1^\dagger] = \frac{1}{2\gamma\omega_1 (\omega_1^2 - \omega_2^2)}, \quad [a_2, a_2^\dagger] = -\frac{1}{2\gamma\omega_2 (\omega_1^2 - \omega_2^2)} \quad (140) \]

To avoid negative norm states, try

\[ a_1|\Omega\rangle = a_2^\dagger|\Omega\rangle = 0, \quad \langle \Omega | a_2^\dagger a_2 | \Omega \rangle > 0, \quad H_{PU} | \Omega \rangle = \frac{1}{2}(\omega_1 - \omega_2)|\Omega\rangle \quad (141) \]

Energy spectrum is unbounded below. So instead try

\[ a_1|\Omega\rangle = a_2|\Omega\rangle = 0, \quad \langle \Omega | a_2 a_2^\dagger | \Omega \rangle < 0, \quad H_{PU} | \Omega \rangle = \frac{1}{2}(\omega_1 + \omega_2)|\Omega\rangle \quad (142) \]

Negative norm state problem looks insurmountable, but.....
QUANTUM MECHANICS IS A GLOBAL THEORY. NEED TO SUPPLY GLOBAL INFORMATION. NEED TO LOOK AT WAVE FUNCTIONS. FIND THAT $H_{PU}$, $z$, $p_z$ ARE NOT HERMITIAN

$$[x, p_x] = i, \quad p_x = -i \frac{\partial}{\partial x}, \quad [z, p_z] = i, \quad p_z = -i \frac{\partial}{\partial z}$$ (143)

$$\psi_0(z, x) = \exp \left[ \frac{\gamma}{2}(\omega_1 + \omega_2)\omega_1 \omega_2 z^2 + i\gamma \omega_1 \omega_2 zx - \frac{\gamma}{2}(\omega_1 + \omega_2)x^2 \right]$$ (144)

The states of negative norm are also states of INFINITE norm since $\int dxdz \psi_0^*(z, x)\psi_0(z, x)$ is divergent, and when acting on such states, one CANNOT set $p_z = -i\partial/\partial z$.

$$\left[ e^{i\theta}z, -\frac{i}{e^{i\theta}} \frac{\partial}{\partial z} \right] \psi(e^{i\theta}z) = i\psi(e^{i\theta}z), \quad z \to -iz, \quad p_z \to \frac{\partial}{\partial z}$$ (145)

$p_z$ and $z$ not Hermitian – they are anti-Hermitian.

$$y = e^{\pi p_z z^2 / 2} z e^{-\pi p_z z^2 / 2} = -iz, \quad q = e^{\pi p_z z^2 / 2} p_z e^{-\pi p_z z^2 / 2} = ip_z$$ (146)

$$H = \frac{p^2}{2\gamma} - iq x + \frac{\gamma}{2} (\omega_1^2 + \omega_2^2) x^2 + \frac{\gamma}{2} \omega_1^2 \omega_2^2 y^2 \neq H^\dagger, \quad p = p_x$$ (147)

Hermitian $[x, p] = i$, Hermitian $[y, q] = i$, non-Hermitian $H$ (148)
The Hamiltonian is not Hermitian – but it is PT symmetric, and thus still has real eigenvalues. Bender and collaborators showed that $H = p^2 + ix^3$ has a completely real energy spectrum.

$$C^2 = 1, \quad [C, PT] = 0, \quad [C, H] = 0, \quad C = e^{Q P} \quad (149)$$

$$Q = \alpha[pq + \gamma^2 \omega_1^2 \omega_2^2 xy], \quad \alpha = \frac{1}{\gamma \omega_1 \omega_2} \log \left( \frac{\omega_1 + \omega_2}{\omega_1 - \omega_2} \right) \quad (150)$$

$$\tilde{H} = e^{-Q/2} H e^{Q/2} = \frac{p^2}{2 \gamma} + \frac{q^2}{2 \gamma \omega_1^2} + \frac{\gamma}{2} \omega_1^2 x^2 + \frac{\gamma}{2} \omega_1^2 \omega_2^2 y^2 \quad (151)$$

Hamiltonian can be diagonalized by a similarity transformation which is non-unitary since $Q$ is Hermitian rather than anti-Hermitian. Original Hamiltonian $H$ is thus a Hermitian Hamiltonian as written in a skew basis. The eigenstates of $H$ and $\tilde{H}$ are not unitarily equivalent, and thus ....
\[\tilde{H} = e^{-Q/2}He^{Q/2} = \frac{p^2}{2\gamma} + \frac{q^2}{2\gamma\omega_1^2} + \frac{\gamma\omega_1^2 x^2}{2} + \frac{\gamma\omega_1^2 \omega_2^2 y^2}{2}\] (152)

\[\tilde{H}|\tilde{n}\rangle = E_n|\tilde{n}\rangle, \quad H|n\rangle = E_n|n\rangle, \quad |n\rangle = e^{Q/2}|\tilde{n}\rangle\] (153)

\[\langle \tilde{n}|\tilde{H}|\tilde{n}\rangle = E_n\langle \tilde{n}|, \quad \langle n| = \langle \tilde{n}|e^{Q/2}, \quad \langle n|e^{-Q}H = \langle n|e^{-Q}E_n\] (154)

The energy eigenbra \(\langle n|e^{-Q} = \langle n|PC\) is not the Dirac conjugate of the energy eigenket \(|n\rangle\), since \(\langle n|H^\dagger = \langle n|E_n\) is not an eigenvalue equation for \(H\).

\[\langle \tilde{n}|\tilde{m}\rangle = \delta_{m,n}, \quad \Sigma|\tilde{n}\rangle\langle \tilde{n}| = 1, \quad \tilde{H} = \Sigma|\tilde{n}\rangle E_n\langle \tilde{n}|\] (155)

\[\langle n|e^{-Q}|m\rangle = \delta_{m,n}, \quad \Sigma|n\rangle\langle n|e^{-Q} = 1, \quad H = \Sigma|n\rangle E_n\langle n|e^{-Q}\] (156)

The \(e^{-Q}\) norm is positive and so theory is unitary. Since \(C^2 = 1\), its eigenvalues are \(\pm 1\), with the relative plus and minus signs in the fourth-order propagator being due to the fact that the two poles have opposite-signed eigenvalues of \(C\).
10 NON-HERMITICITY AND UNITARITY

\[ i \frac{d}{dt} | R_S(t) \rangle = H | R_S(t) \rangle, \quad -i \frac{d}{dt} \langle R_S(t) | = \langle R_S(t) | H^\dagger \]  

\[ | R_S(t) \rangle = e^{-iHt} | R_S(0) \rangle, \quad \langle R_S(t) | = \langle R_S(0) | e^{iHt} \]  

\[ \langle R_S(t) | R_S(t) \rangle = \langle R_S(0) | e^{iHt} e^{-iHt} | R_S(0) \rangle \neq \langle R_S(0) | R_S(0) \rangle \quad \text{NOT UNITARY} \]  

\[ \langle R_S(t) | A_S | R_S(t) \rangle = \langle R_S(0) | e^{iHt} A_S e^{-iHt} | R_S(0) \rangle = \langle R_S(0) | A_H(t) | R_S(0) \rangle \]  

\[ A_H(t) = e^{iHt} A_S e^{-iHt}, \quad \text{OBTAIN} \quad i \frac{d}{dt} A_H(t) = A_H(t) H - H^\dagger A_H(t) \]  

\[ i \frac{d}{dt} | R_S(t) \rangle = H | R_S(t) \rangle, \quad -i \frac{d}{dt} \langle L_S(t) | = \langle L_S(t) | H, \quad | R_S(t) \rangle = e^{-iHt} | R_S(0) \rangle, \quad \langle L_S(t) | = \langle L_S(0) | e^{iHt} \]  

\[ A_S = e^{-iHt} A_H(t) e^{iHt}, \quad \langle L_S(t) | A_S | R_S(t) \rangle = \langle L_S(0) | e^{iHt} A_S e^{-iHt} | R_S(0) \rangle = \langle L_S(0) | A_H(t) | R_S(0) \rangle \]  

\[ \langle L_S(t) | R_S(t) \rangle = \langle L_S(0) | e^{iHt} e^{-iHt} | R_S(0) \rangle = \langle L_S(0) | R_S(0) \rangle \quad \text{UNITARY} \]  

LEFT-EIGENVECTOR IS NOT DIRAC CONJUGATE OF RIGHT EIGENVECTOR

\[ H^\dagger = e^{-Q} H e^Q, \quad \langle R | H^\dagger = \langle R | E, \quad \langle R | e^{-Q} H = \langle R | e^{-Q} E, \quad \langle L | = \langle R | e^{-Q}, \quad \sum |R\rangle \langle L | = \sum |R\rangle \langle R | e^{-Q} = 1 \]  

\[ [e^{-Q} H]^\dagger = H^\dagger e^{-Q} = e^{-Q} H, \quad H \text{ is not Hermitian but is SELF-ADJOINT} \]
NON-DIAGONALIZABILITY AND UNITARITY
THE SINGULAR EQUAL-FREQUENCY LIMIT

In equal frequency limit the diagonalizing operator $Q$ becomes singular and partial fraction decomposition of propagator becomes undefined.

$$\psi_0(x, y, t) = \exp \left[ -\frac{\gamma}{2}(\omega_1 + \omega_2)(x^2 + \omega_1 \omega_2 y^2) - \gamma \omega_1 \omega_2 y x \right] \exp(-iE_0 t),$$

$$E_0 = (\omega_1 + \omega_2)/2$$

$$\psi_1(x, y, t) = (x + \omega_2 y)\psi_0(x, y, t)e^{-i\omega_1 t}, \quad E_1 = E_0 + \omega_1$$

$$\psi_2(x, y, t) = (x + \omega_1 y)\psi_0(x, y, t)e^{-i\omega_2 t}, \quad E_2 = E_0 + \omega_2$$

$$\hat{\psi}_0(x, y, t) = \exp \left[ -\gamma \omega^3 y^2 - \gamma \omega^2 y x - \gamma \omega x^2 - i\omega t \right], \quad \hat{E}_0 = \omega$$

$$\hat{\psi}_1(x, y, t) = (x + \omega y)\hat{\psi}_0(x, y, t)e^{-i\omega t}, \quad \hat{E}_1 = \hat{E}_0 + \omega$$

TWO one-particle states have collapsed into ONE state.
SAME FOR THE HIGHER MULTI-PARTICLE STATES

\[ \psi_0(x, y, t) = \exp \left[ -\frac{\gamma}{2} (\omega_1 + \omega_2) (x^2 + \omega_1 \omega_2 y^2) - \gamma \omega_1 \omega_2 y x \right] \exp(-iE_0t), \quad E_0 = (\omega_1 + \omega_2)/2 \quad (172) \]

\[
\begin{align*}
\psi_1(x, y, t) &= (x + \omega_2 y) \psi_0(x, y, t) e^{-i\omega_1 t}, \quad E_1 = E_0 + \omega_1 \\
\psi_2(x, y, t) &= (x + \omega_1 y) \psi_0(x, y, t) e^{-i\omega_2 t}, \quad E_2 = E_0 + \omega_2
\end{align*}
\]

(173)

\[
\begin{align*}
\psi_3(x, y, t) &= \left[ (x + \omega_2 y)^2 - 1/2\gamma \omega_1 \right] \psi_0(x, y, t) e^{-2i\omega_1 t}, \quad E_3 = E_0 + 2\omega_1 \\
\psi_4(x, y, t) &= \left[ (x + \omega_1 y)(x + \omega_2 y) - 1/\gamma (\omega_1 + \omega_2) \right] \psi_0(x, y, t) e^{-i(\omega_1 + \omega_2)t}, \quad E_4 = E_0 + \omega_1 + \omega_2 \\
\psi_5(x, y, t) &= \left[ (x + \omega_1 y)^2 - 1/2\gamma \omega_2 \right] \psi_0(x, y, t) e^{-2i\omega_2 t}, \quad E_5 = E_0 + 2\omega_2
\end{align*}
\]

(174)

\[ \hat{\psi}_0(x, y, t) = \exp \left[ -\gamma \omega_3 y^2 - \gamma \omega_2 y x - \gamma \omega x^2 - i\omega t \right], \quad \hat{E}_0 = \omega \quad (175) \]

\[ \hat{\psi}_1(x, y, t) = (x + \omega y) \hat{\psi}_0(x, y, t) e^{-i\omega t}, \quad \hat{E}_1 = \hat{E}_0 + \omega \quad (176) \]

\[ \hat{\psi}_2(x, y, t) = \left[ (x + \omega y)^2 - 1/2\gamma \omega \right] \hat{\psi}_0(x, y, t) e^{-2i\omega t}, \quad \hat{E}_2 = \hat{E}_0 + 2\omega \quad (177) \]
THE MISSING ENERGY EIGENSTATES....

\[ H_{1P}(\epsilon) = \frac{1}{2\omega} \left( \begin{array}{cc} 4\omega^2 + \epsilon^2 & 4\omega^2 - \epsilon^2 \\ \epsilon^2 & 4\omega^2 - \epsilon^2 \end{array} \right) \], \quad \omega_1 \equiv \omega + \epsilon, \quad \omega_2 \equiv \omega - \epsilon \] (178)

\[ |2\omega + \epsilon\rangle \equiv \begin{pmatrix} 2\omega + \epsilon \\ \epsilon \end{pmatrix}, \quad |2\omega - \epsilon\rangle \equiv \begin{pmatrix} 2\omega - \epsilon \\ -\epsilon \end{pmatrix} \] (179)

\[ S^{-1} \left( \frac{1}{2\omega} \right) \left( \begin{array}{cc} 4\omega^2 + \epsilon^2 & 4\omega^2 - \epsilon^2 \\ \epsilon^2 & 4\omega^2 - \epsilon^2 \end{array} \right) S = \begin{pmatrix} 2\omega + \epsilon & 0 \\ 0 & 2\omega - \epsilon \end{pmatrix} \] (180)

\[ S' = \frac{1}{2\epsilon \omega^{1/2}(2\omega + \epsilon)^{1/2}} \begin{pmatrix} 2\omega + \epsilon & -(4\omega^2 - \epsilon^2)\epsilon \\ \epsilon & (2\omega + \epsilon)\epsilon^2 \end{pmatrix} \] (181)

\[ H_{1P}(\epsilon = 0) = 2\omega \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \] (182)

Non-diagonalizable, Jordan-block matrix with TWO eigenvalues (\(\lambda_1 = 1, \lambda_2 = 1\) since \(\text{Tr} = 1, \text{Det} = 1\)), but only ONE eigenvector.

\[ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} c + d \\ d \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}, \quad d = 0 \] (183)
... BECAME NONSTATIONARY

\[ \hat{\psi}_1(x, y, t) = \lim_{\epsilon \to 0} \frac{\psi_2(x, y, t) + \psi_1(x, y, t)}{2} = (x + \omega y)\hat{\psi}_0(x, y, t)e^{-i\omega t} \]

\[ \hat{\psi}_{1a}(x, y, t) = \lim_{\epsilon \to 0} \frac{\psi_2(x, y, t) - \psi_1(x, y, t)}{2\epsilon} = [(x + \omega y)it + y] \hat{\psi}_0(x, y, t)e^{-i\omega t} \]

\[ \hat{\psi}_2(x, y, t) = \lim_{\epsilon \to 0} \frac{\psi_3(x, y, t) + \psi_4(x, y, t) + \psi_5(x, y, t)}{3} = [(x + \omega y)^2 - 1/2\gamma \omega] \hat{\psi}_0(x, y, t)e^{-2i\omega t} \]

\[ \hat{\psi}_{2a}(x, y, t) = \lim_{\epsilon \to 0} \frac{\psi_5^R(x, y, t) - \psi_3^R(x, y, t)}{2\epsilon} \]

\[ = \left[ \left( (x + \omega y)^2 - \frac{1}{2\gamma \omega} \right) 2it + 2xy + 2\omega y^2 - \frac{1}{2\gamma \omega^2} \right] \hat{\psi}_0^R(x, y, t)e^{-2i\omega t} \]

\[ \hat{\psi}_{2b}(x, y, t) = \lim_{\epsilon \to 0} \frac{2\psi_4^R(x, y, t) - 3\psi_3^R(x, y, t) - \psi_5^R(x, y, t)}{2\epsilon^2} \]

\[ = \left[ \left( (x + \omega y)^2 - \frac{1}{2\gamma \omega} \right) 2t^2 - \left( 2xy + 2\omega y^2 - \frac{1}{2\gamma \omega^2} \right) 2it - 2y^2 + \frac{1}{2\gamma \omega^3} \right] \hat{\psi}_0^R(x, y, t)e^{-2i\omega t} \]  

(184)

\[ i\frac{\partial}{\partial t} \hat{\psi}(x, y, t) = \left( -\frac{1}{2\gamma} \frac{\partial^2}{\partial x^2} - x \frac{\partial}{\partial y} + \gamma \omega^2 x^2 + \frac{\gamma}{2} \omega^4 y^2 \right) \hat{\psi}(x, y, t) \]  

(185)

Stationary plus non-stationary together are complete since just the right number of independent polynomial functions of \( x \) and \( y \).

\[ i\frac{\partial}{\partial t} \int dxdy \hat{\psi}_B^c(x, y, t)\hat{\psi}_A(x, y, t) = -\int dxdy x \frac{\partial}{\partial y} \left[ \hat{\psi}_B^c(x, y, t)\hat{\psi}_A(x, y, t) \right] = 0 \]  

(186)

Norm preserved in time so time evolution is UNITARY.
Conformal Supergravity in Twistor-String Theory

N. Berkovits and E. Witten


“The net effect is that the translation generator $D$ acts as

$$
\begin{pmatrix}
P & * \\
0 & P
\end{pmatrix}
$$

where $P$ would represent ordinary translations and the off-diagonal $*$ arises from $[D, \partial_I] \neq 0$.

This matrix is not diagonalizable. This clashes with our usual experience. We are accustomed to the idea that the translation generators are Hermitian operators and so can be diagonalized. However, conformal supergravity is not a unitary theory, and one symptom of this is that the translation generators are undiagonalizable.”

========

Bender and Mannheim: Not so fast.
WHAT IS SO SPECIAL ABOUT FOURTH-ORDER THEORIES TO CAUSE ALL THIS

Consider the Lehmann Representation for a scalar field. Assume translation invariance and a bounded, real energy eigenspectrum with states of 4-momentum $k_n^\mu$ with $k_n^0 > 0$. We can set $\phi(x) = e^{iP^\mu x}\phi(0)e^{-iP^\mu x}$. Provisionally set $\Sigma \langle n|n\rangle = 1$. Then can show

$$\langle \Omega|\phi(x)\phi(y)|\Omega\rangle = \sum_n \langle \Omega|\phi(0)|k_n^\mu\rangle^2 e^{-ik_n^\mu(x-y)}$$

(187)

Now introduce the spectral function

$$\rho(q^2) = (2\pi)^3 \sum_n \delta^4(k_n^\mu - q^\mu)\langle \Omega|\phi(0)|k_n^\mu\rangle^2 \theta(q_0)$$

(188)

and set

$$\langle \Omega|\phi(x)\phi(y)|\Omega\rangle = \int_0^\infty dm^2 \rho(m^2) \int \frac{d^4q}{(2\pi)^3} \theta(q_0) \delta(q^2 - m^2) e^{-iq(x-y)}$$

(189)

Thus finally obtain the Lehmann representation

$$\langle \Omega|T[\phi(x)\phi(y)]|\Omega\rangle = \int_0^\infty dm^2 \rho(m^2) \Delta^\text{free}_2(x - y; m^2)$$

(190)

This relation holds for any interacting two-point function no matter what its equation of motion, with it always being the FREE SECOND-ORDER Feynman propagator which appears in the integral because the mass shell condition $p^2 = m^2$ is always second-order. The Lehmann representation thus holds in fourth-order theories also. However, for large $k^2$ the second-order Feynman propagator behaves as $1/k^2$, whereas for fourth-order theories the propagator behaves as $1/k^4$. Hence we have a contradiction if $\rho(m^2)$ is positive definite.
Solution is that spectral function cannot be positive definite, and we cannot set $\sum |n\rangle\langle n| = 1$. Rather we must set $\sum |n\rangle\langle n| e^{-Q} = 1$ (and not $\sum |n\rangle\langle n| - \sum |m\rangle\langle m| = 1$), and distinguish between left and right momentum eigenvectors. Thus must use $|R\rangle$ and $\langle L| = \langle R| e^{-Q}$.

With this choice, the spectral function is replaced by

$$\rho(q^2) = (2\pi)^3 \sum_n \delta^4(k_\mu^n - q_\mu) \langle \Omega| e^{-Q} \phi(0)|k_\mu^n\rangle \langle k_\mu^n| e^{-Q} \phi(0)|\Omega\rangle \theta(q_0)$$

(191)

Now there is no positivity requirement and we can cancel the $1/k^2$ behavior without having to give up unitarity. Since we still impose the reality of the momentum eigenvalues, the Hamiltonian of the theory must be $PT$ invariant rather than Hermitian. From the Lehmann representation we thus conclude that the Hamiltonian of theories such as the fourth-order Pais-Uhlenbeck oscillator cannot be Hermitian and must instead be $PT$ invariant, just as we had found directly.

**THE REMARKABLE MORAL OF THE STORY**

Consider **ANY** higher derivative theory in which the equation of motion is of the form $f(D)\phi = 0$ where $D = \partial_\mu \partial^\mu$ and $f(D) = \sum a_n D^n$. Also require that all momentum eigenvalues be real. In such a theory, at large $k^2$ the propagator will behave as $1/k^{2n}$. Hence there will be a contradiction with the Lehmann representation if we require the standard $\sum |n\rangle\langle n| = 1$. Rather, such theories must be $PT$ theories rather than standard Hermitian ones. Hence once we depart from second-order equations we are forced to $PT$-invariant, non-Hermitian Hamiltonians. $PT$-invariance is thus the general rule, and it is only a historical accident (second-order theories were encountered first) that it was not discovered earlier.
GENERAL ASPECTS OF PT SYMMETRY

The following PT theorems have now been established:

(I) If \([H, PT] = 0\) the secular equation \(\det(H - EI) = 0\) which determines the energy eigenvalues is real. Thus energies are real or appear in complex conjugate pairs.

(II) If the secular equation is real, then \([H, PT] = 0\).

Thus if \([H, PT] \neq 0\), not all of the energy eigenvalues can be real.

Contrast: If \(H \neq H^\dagger\) no constraint and energies could still all be real.

Introduce \(C\) operator which obeys \([C, H] = 0, C^2 = 1\).

(III) If \([C, PT] = 0\) all of the energy eigenvalues are real.

(IV) If \([C, PT] \neq 0\) some energy eigenvalues are complex.

Thus \([H, PT] = 0\) and \([C, PT] = 0\) is NECESSARY and SUFFICIENT condition for reality of energy eigenvalues.
DYNAMICAL SYMMETRY BREAKING
AND THE COSMOLOGICAL CONSTANT PROBLEM

The zero-point energy and cosmological constant problems are two separate problems. With dynamical symmetry breaking they solve each other.

The zero-point energy problem already exists in a free field theory, and as such is separate from any cosmological constant term that might be induced by spontaneous symmetry breaking. The cosmological constant is associated with the minimum of the symmetry breaking potential while the zero-point energy is associated with the fluctuations about it. Moreover, the zero-point term and the cosmological constant term even transform differently under a general coordinate transformation, the former possessing a fluid velocity and being maximally 3-symmetric, with the latter possessing no fluid velocity and being maximally 4-symmetric. The zero-point fluctuation term is associated with a perfect matter fluid in which both $\rho_m$ and $p_m$ are positive, so that $T_{\mu\nu} = (\rho_m + p_m)U_{\mu}U_{\nu} + p_m g_{\mu\nu}$. While the cosmological constant term is associated with a perfect fluid in which $p = -\rho$, so that $T_{\mu\nu} = -\Lambda g_{\mu\nu}$.

When the symmetry is broken dynamically by fermion condensates, it is the fermionic zero-point fluctuations which cause the change in the vacuum in the first place, to thus actually produce the cosmological constant. In this case the zero-point and cosmological constant terms are not independent, and are related in a way which allows each one to cancel the other, so that both the zero-point and cosmological constant problems solve each other.
For a free quantum fermion field, in a mode of the form

\[ \psi(x) = \sum_{\pm s} \int \frac{d^4 p}{(2\pi)^{3/2}} \left( \frac{m}{E_p} \right)^{1/2} \left[ b(p, s) u(p, s) e^{-ip \cdot x} + d^\dagger(p, s) v(p, s) e^{ip \cdot x} \right] \]  

the vacuum expectation value of \( T_{\mu\nu} \sim \bar{\psi} \gamma_\mu \partial_\nu \psi \) is due to the non-vanishing of

\[ \langle \Omega | d(p, s) d^\dagger(p, s) | \Omega \rangle = \langle \Omega | (d(p, s) d^\dagger(p, s) + d^\dagger(p, s) d(p, s)) | \Omega \rangle, \]

(193)

to yield the zero-point fluctuation contribution

\[ \langle \Omega | T_{\mu\nu} | \Omega \rangle \sim -2 \int \frac{d^3 p}{(2\pi)^3} \frac{p_\mu p_\nu}{E_p}, \quad \rho_m = \langle \Omega | T_{00} | \Omega \rangle \sim -2 \int \frac{d^3 p}{(2\pi)^3} E_p, \]

\[ p_m = \langle \Omega | T_{xx} | \Omega \rangle = \langle \Omega | T_{yy} | \Omega \rangle = \langle \Omega | T_{zz} | \Omega \rangle \sim -2 \int \frac{d^3 p}{(2\pi)^3} \frac{p^2}{3 E_p}. \]

(194)

and a perfect fluid form \( T_{\mu\nu} = (\rho_m + p_m) U_\mu U_\nu + p_m g_{\mu\nu} \) where \( \rho_m \) and \( p_m \) have the SAME sign. That this is NOT a perfect fluid with \( p_m = -\rho_m \) is because the averaging is only 3-dimensional, i.e. for modes which obey the wave equation only the 3-momentum is averaged over. Moreover, the velocity of the fluid \( U^\mu \) is a timelike vector, and is allowed simply because a Minkowski metric distinguishes between timelike and spacelike.

In flat space, for \( T_{\mu\nu} \sim k_\mu k_\nu / E_k \) one can also construct the total energy and momentum as \( P_\mu = \int d^3 x T_{0\mu} \), to find that

\[ P_0 \sim E_k, \quad P_i = 0, \]

(195)
with there being no zero-point total momentum. However one can also evaluate

\[ \int d^3 x T_{ij} \sim \int d^3 x \frac{k_i k_j}{E_k} \sim \delta_{ij} \]  

(196)
to thus yield a zero-point quadrupole pressure tensor. Thus the absence of a zero-point total momentum does not mean the absence of a zero-point pressure; and while such a term plays no role in flat space it still couples to gravity in curved space.

To underscore the fact that the zero-point term is not the same as a cosmological constant term, we note that for a free massless field where \( p^\mu p_\mu = 0 \), the energy-momentum tensor is traceless and \( \rho_m = 3p_m \), something not possible for a cosmological constant term, since a traceless \( \Lambda g_{\mu\nu} \) would require \( \Lambda = 0 \).

A cosmological constant term is induced when the symmetry is broken and adds a term of the form of \( T_{\mu\nu} = -\Lambda g_{\mu\nu} \). It yields a contribution to the trace of the form \( T^\mu_\mu = -4\Lambda \), with the zero-point energy and the cosmological constant terms thus being different. In the presence of both the energy density is given by \( T_{00} = \rho_m + \Lambda \).

When mass is generated dynamically in a four-Fermi theory, the mean field is produced by zero-point fluctuations. The cosmological constant is thus induced by zero-point fluctuations also and can thus be related to the zero-point energy. Since the trace of \( T_{\mu\nu} \) is zero in a conformal invariant theory, \( 3p_m - \rho_m - 4\Lambda = 0 \), and the zero-point and cosmological constant terms cancel each other identically. Neither can be bigger or smaller than the other.
FOUR-FERMI MEAN-FIELD THEORY IN TWO DIMENSIONS

In $D = 2$ the four-Fermi theory is defined via

$$L = i\bar{\psi}\gamma^\mu \partial_\mu \psi - (g/2)[\bar{\psi}\psi]^2, \quad i\gamma^\mu \partial_\mu \psi - g[\bar{\psi}\psi]\psi = 0, \quad (197)$$

$$T^{\mu\nu} = i\bar{\psi}\gamma^\mu \partial_\nu \psi - g^{\mu\nu} \frac{g}{2}[\bar{\psi}\psi]^2, \quad T^\mu_\mu = i\bar{\psi}\gamma^\mu \partial_\mu \psi - g[\bar{\psi}\psi]^2 = 0, \quad (198)$$

with the energy-momentum tensor being traceless. In the mean-field approximation one looks for self-consistent, translation invariant, states $|S\rangle$ in which

$$\langle S|\bar{\psi}\psi|S\rangle = m/g, \quad \langle S|[\bar{\psi}\psi - m/g]^2|S\rangle = 0, \quad i\gamma^\mu \partial_\mu \psi - m\psi = 0, \quad (199)$$

$$\langle S|T^{\mu\nu}|S\rangle = \langle S|i\bar{\psi}\gamma^\mu \partial_\nu \psi|S\rangle - g^{\mu\nu} \frac{m^2}{2g}, \quad \langle S|T^\mu_\mu|S\rangle = m\langle S|\bar{\psi}\psi|S\rangle - \frac{m^2}{g} = 0. \quad (200)$$

with the mean-field approximation preserving tracelessness.

In a plane wave solution

$$\psi(x) = \sum_{\pm s} \int d^2p/(2\pi)^{1/2}(m/E_p)^{1/2} [b(p, s)u(p, s)e^{-ip\cdot x} + d^{\dagger}(p, s)v(p, s)e^{ip\cdot x}] \quad (201)$$

we find that

$$\langle \Omega|i\bar{\psi}\gamma^\mu \partial_\nu \psi|\Omega\rangle = -2 \int d^2pp^\mu p^\nu/(2\pi E_p) \quad (202)$$

due to zero-point fluctuations since

$$\langle \Omega|bb^{\dagger}|\Omega\rangle = \langle \Omega|(bb^{\dagger} + b^{\dagger}b)|\Omega\rangle. \quad (203)$$
CANCELLATION OF ZERO-POINT AND $\Lambda$ IN TRACE OF $T_{\mu\nu}$

Incoherently adding together modes with $p^\mu = (E_p, p)$ and $(E_p, -p)$ gives

$$\sum \frac{p^\mu p^\nu}{E_p} = \begin{pmatrix} E_p & p \\ p & p^2/E_p \end{pmatrix} + \begin{pmatrix} E_p & -p \\ -p & p^2/E_p \end{pmatrix} = \begin{pmatrix} 2E_p & 0 \\ 0 & 2p^2/E_p \end{pmatrix} = (\rho_m + p_m)U^{\mu}U^{\nu} + p_m g^{\mu\nu}$$

On setting $m^2/g = g[\langle S|\bar{\psi}\psi|S\rangle]^2 = \Lambda$, we can thus set

$$\langle S|T^{\mu\nu}|S\rangle = (\rho_m + p_m)U^{\mu}U^{\nu} + p_m g^{\mu\nu} - g^{\mu\nu}\Lambda, \quad \langle S|T^\mu_\mu|S\rangle = p_m - \rho_m - 2\Lambda = 0. \quad (205)$$

Thus $2\Lambda = p_m - \rho_m$, neither bigger nor smaller. The reason for this is that all of $\Lambda$, $\rho_m$ and $p_m$ are determined in one the same state $|S\rangle$. For the case of a fundamental scalar field $\phi(x)$, $\Lambda = \lambda \phi^4$ is determined by the vacuum (location of the minimum of the Higgs double-well potential), while $\rho_m$ and $p_m$ are determined by whichever matter field frequency modes are occupied. Hence one cannot relate $\Lambda$ to $\rho_m$ and $p_m$ in standard cosmology with a fundamental Higgs field, to thus give rise to the cosmological constant problem. If however scalar field is a c-number Ginzburg-Landau condensate $\langle S|\bar{\psi}\psi|S\rangle$ then can relate relate $\Lambda$, $\rho_m$ and $p_m$, and even have them cancel each other.

**Comparison test:** If Higgs is a c-number condensate then will NOT be produced in an accelerator such as the LHC, while if Higgs is fundamental then will be produced in an accelerator.


CANCELLATION OF ZERO-POINT AND $\Lambda$ IN $T_{\mu\nu}$ ITSELF

In four spacetime dimensions invariance under

$$g_{\mu\nu}(x) \rightarrow e^{2\alpha(x)}g_{\mu\nu}(x),$$

leads to a unique gravitational action

$$I_W = -\alpha_g \int d^4x (-g)^{1/2} C_{\lambda\mu\nu\kappa} C^{\lambda\mu\nu\kappa} = -2\alpha_g \int d^4x (-g)^{1/2} \left[ R^{\mu\nu} R_{\mu\nu} - \frac{1}{3} R^\alpha \alpha \right]$$

where $\alpha_g$ is dimensionless and $C_{\lambda\mu\nu\kappa}$ is the conformal Weyl tensor. The associated gravitational equations of motion are the fourth-order derivative:

$$-4\alpha_g \left[ 2C^{\mu\lambda\nu\kappa}_{;\lambda;\kappa} - C^{\mu\lambda\nu\kappa} R_{\lambda\kappa} \right] + T^{\mu\nu} = 0$$

$$T^{\mu\nu}_{\text{total}} = T^{\mu\nu}_{\text{grav}} + T^{\mu\nu}_{\text{matter}} = 0.$$ 

While $\langle S | \left[ 2C^{\mu\lambda\nu\kappa}_{;\lambda;\kappa} - C^{\mu\lambda\nu\kappa} R_{\lambda\kappa} \right] | S \rangle$ vanishes in a classical cosmological background, quantum-mechanically it contains a zero-point gravitational fluctuation contribution, and because of the gravitational equation of motion the gravitational zero-point contribution precisely cancels the zero-point and $\Lambda$ contributions to $T^{\mu\nu}$. Using the gravitational zero-point fluctuations to cancel the matter field zero-point fluctuations and $\Lambda$ is only achievable in a renormalizable theory of gravity. Hence works in conformal gravity but not in standard gravity.
\begin{equation}
\text{STANDARD COSMOLOGY}
\end{equation}

\begin{equation}
ds^2 = -c^2 dt^2 + R^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right]
\end{equation}

\begin{equation}
H = \frac{\dot{R}}{R}, \quad q = -\frac{R\ddot{R}}{R^2}
\end{equation}

\begin{equation}
T_{\mu\nu} = (\rho_M + p_M)U_{\mu}U_{\nu} + p_M g_{\mu\nu} - \Lambda g_{\mu\nu}, \quad \rho_M = \frac{A}{R^n}
\end{equation}

\begin{equation}
\dot{R}^2 + kc^2 = R^2 [\Omega_M + \Omega_\Lambda]
\end{equation}

\begin{equation}
q = -\frac{R\ddot{R}}{R^2} = \left( \frac{n}{2} - 1 \right) \Omega_M - \Omega_\Lambda
\end{equation}

\begin{equation}
\Omega_M = \frac{8\pi G \rho_M}{3c^2 H^2}, \quad \Omega_\Lambda = \frac{8\pi G \Lambda}{3c H^2},
\end{equation}

\begin{equation}
\frac{\Omega_\Lambda}{\Omega_M} = \frac{c\Lambda}{\rho_M} = \frac{T_V^4}{T^4}
\end{equation}
Calan/Tololo
(Hamuy et al, A.J. 1996)

Supernova Cosmology Project

$\Omega_M, \Omega_\Lambda = (0, 1)$

$\Omega_M, \Omega_\Lambda = (0.5, 0.5)$

$\Omega_M, \Omega_\Lambda = (1, 0)$

$\Omega_M, \Omega_\Lambda = (1.5, -0.5)$

$\Omega_M, \Omega_\Lambda = (2, 0)$

Flat

$\Omega_\Lambda = 0$

$\Omega_M, \Omega_\Lambda = (0, 0)$

Calan/Tololo (Hamuy et al, A.J. 1996)

Effective $m_B$

Redshift $z$

$z = 0.02, 0.05, 0.1, 0.2, 0.5, 1.0$

$0.02, 0.05, 0.1, 0.2, 0.5, 1.0$
No Big Bang

\[ \Lambda = 0 \]

Universe

Flat

-1

0

1

2

3

\( \Omega_{\Lambda} \)

\( \Omega_{\text{M}} \)

expands forever

recollapses eventually

closed

flat

open

68%

90%

95%

99%
CONFORMAL COSMOLOGY

\[ I_M = -\int d^4x (-g)^{1/2}\left[ \frac{1}{2} S^{\mu\nu} S_{\mu\nu} - \frac{1}{12} S^2 R_{\mu} R_{\mu} + \lambda S^4 + i\bar{\psi}\gamma^\mu[\partial_{\mu} + \Gamma_{\mu}]\psi - hS\bar{\psi}\psi \right] \]  

(217)

\[ T_{\mu\nu} = (\rho_M + p_M)U_{\mu}U_{\nu} + p_M g_{\mu\nu} - \frac{1}{6} S_{0}^2 \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R^\alpha_{\alpha} \right) - g_{\mu\nu} \lambda S_{0}^4 \]  

(218)

\[ C_{\lambda\mu\nu\kappa} = 0, \quad T_{\mu\nu} = 0 \]  

(219)

\[ \frac{1}{6} S_{0}^2 \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R^\alpha_{\alpha} \right) = (\rho_M + p_M)U_{\mu}U_{\nu} + p_M g_{\mu\nu} - g_{\mu\nu} \lambda S_{0}^4 \]  

(220)

\[ G_{\text{eff}} = -\frac{3}{4\pi S_{0}^2} \]  

(221)

\[ \dot{R}^2 + kc^2 = \dot{R}^2 \left[ \bar{\Omega}_M + \bar{\Omega}_\Lambda \right], \quad q = -\frac{R\ddot{R}}{R^2} = \left( \frac{n}{2} - 1 \right) \bar{\Omega}_M - \bar{\Omega}_\Lambda \]  

(222)

\[ \bar{\Omega}_M = \frac{8\pi G_{\text{eff}}\rho_M}{3c^2 H^2}, \quad \bar{\Omega}_\Lambda = \frac{8\pi G_{\text{eff}}\Lambda}{3cH^2}, \quad \frac{\bar{\Omega}_\Lambda}{\bar{\Omega}_M} = \frac{c\Lambda}{\rho_M} = -\frac{T_V^4}{T_4^4} \]  

(223)

\[ \bar{\Omega}_\Lambda = \left( 1 - \frac{T^2}{T_{\text{max}}^2} \right)^{-1} \left( 1 + \frac{T^2 T_{\text{max}}^2}{T_V^4} \right)^{-1}, \quad \bar{\Omega}_M = -\frac{T_V^4}{T_V^4} \bar{\Omega}_\Lambda \]  

(224)

\[ 0 \leq \bar{\Omega}_\Lambda \leq 1, \quad -1 \leq q \leq 0 \]  

(225)

\[ d_L = -\frac{c}{H_0 q_0} \left( 1 + z \right)^2 \left( 1 - \left[ 1 + q_0 - \frac{q_0}{\left( 1 + z^2 \right)} \right]^{1/2} \right) \]  

(226)
Figure 9: The $q_0 = -0.37$ conformal gravity fit (upper curve) and the $\Omega_M(t_0) = 0.3, \Omega_\Lambda(t_0) = 0.7$ standard model fit (lower curve) to the $z < 1$ supernovae Hubble plot data.
Figure 10: Hubble plot expectations for $q_0 = -0.37$ (highest curve) and $q_0 = 0$ (middle curve) conformal gravity and for $\Omega_M(t_0) = 0.3$, $\Omega_\Lambda(t_0) = 0.7$ standard gravity (lowest curve).
EVEN IF NON-NORMALIZABLE, STATES CAN STILL BE COMPLETE

Completeness of non-normalizable modes

Philip D. Mannheim and Ionel Simbotin


\[
\int_{-\infty}^{\infty} dwe^{-2A(w)} f_m(w) f_{m'}(w) = \delta_{m,m'}
\]  \hspace{1cm} (227)

\[
\psi(w) = \sum_m a_m f_m(w)
\]  \hspace{1cm} (228)

\[
a_m = \int_{-\infty}^{\infty} dwe^{-2A(w)} f_m(w) \psi(w)
\]  \hspace{1cm} (229)

\[
\psi(w) = \sum_m \int_{-\infty}^{\infty} dw' e^{-2A(w')} f_m(w') f_m(w) \psi(w')
\]  \hspace{1cm} (230)

\[
\psi(w) = \int_{-\infty}^{\infty} dw' \delta(w - w') \psi(w')
\]  \hspace{1cm} (231)

\[
\sum_m f_m(w') f_m(w) = e^{2A(w)} \delta(w - w').
\]  \hspace{1cm} (232)
Reconstruction of the square step $V_J(|w|) = 1$, $1 < |w| < 2$, $V_J = 0$ otherwise via sum $V_J(|w|) = \sum a_i J_2(j_i e^{-|w|})$ on CONVERGENT NORMALIZABLE modes with basis states which obey $J_1(j_i) = 0$. 

\[ V_J(w) \]

\[ J_1(j_i) \]

\[ e^{-w} \]
Reconstruction of the square step $V_Y(|w|) = 1$, $1 < |w| < 2$, $V_Y = 0$ otherwise via sum $V_Y(|w|) = \sum a_i Y_2(y_i e^{-|w|})$ on DIVERGENT NON-NORMALIZABLE modes with basis states which obey $Y_1(y_i) = 0$.

CONCLUSION: All that matters is LINEAR relation: $\psi(w) = \sum m a_m f_m(w)$. No need to require BILINEAR relation $\sum m f_m(w')f_m(w) = e^{2A(w)} \delta(w - w')$.

IMPLICATION: $H|\psi\rangle = E|\psi\rangle$ is linear. No reference to $\langle \psi | \psi \rangle$. Thus left eigenvector which obeys $\langle L | H = \langle L | E$ need not be conjugate of right eigenvector which obeys $H | R \rangle = E | R \rangle$. 
WHAT IS SO SPECIAL ABOUT FOURTH-ORDER THEORIES TO CAUSE ALL THIS

Consider the Lehmann Representation for a scalar field. Assume translation invariance and a bounded, real energy eigenspectrum with states of 4-momentum $k^\mu_n$ with $k^0_n > 0$. We can set $\phi(x) = e^{iP\cdot x} \phi(0) e^{-iP\cdot x}$. Provisionally set $\sum |n\rangle \langle n| = 1$. Then can show

$$\langle \Omega | \phi(x) \phi(y) | \Omega \rangle = \sum_n |\langle \Omega | \phi(0) |k^\mu_n \rangle|^2 e^{-ik^\mu_n \cdot (x-y)} \tag{233}$$

Now introduce the spectral function

$$\rho(q^2) = (2\pi)^3 \sum_n \delta^4(k^\mu_n - q^\mu) |\langle \Omega | \phi(0) |k^\mu_n \rangle|^2 \theta(q_0) \tag{234}$$

and set

$$\langle \Omega | \phi(x) \phi(y) | \Omega \rangle = \int_0^\infty dm^2 \rho(m^2) \int \frac{d^4q}{(2\pi)^3} \theta(q_0) \delta(q^2 - m^2) e^{-iq \cdot (x-y)} \tag{235}$$

Thus finally obtain the Lehmann representation

$$\langle \Omega | T[\phi(x) \phi(y)] | \Omega \rangle = \int_0^\infty dm^2 \rho(m^2) \Delta_2^{\text{free}}(x - y; m^2) \tag{236}$$

This relation holds for any interacting two-point function no matter what its equation of motion, with it always being the FREE SECOND-ORDER Feynman propagator which appears in the integral because the mass shell condition $p^2 = m^2$ is always second-order. The Lehmann representation thus holds in fourth-order theories also. However, for large $k^2$ the second-order Feynman propagator behaves as $1/k^2$, whereas for fourth-order theories the propagator behaves as $1/k^4$. Hence we have a contradiction if $\rho(m^2)$ is positive definite.
Solution is that spectral function cannot be positive definite, and we cannot set $\sum |n\rangle \langle n| = 1$. Rather we must set $\sum |n\rangle \langle n| e^{-Q} = 1$ (and not $\sum |n\rangle \langle n| - \sum |m\rangle \langle m| = 1$), and distinguish between left and right momentum eigenvectors. Thus must use $|R\rangle$ and $\langle L| = \langle R| e^{-Q}$. With this choice, the spectral function is replaced by

$$\rho(q^2) = (2\pi)^3 \sum_n \delta^4(k^\mu_n - q^\mu) \langle \Omega| e^{-Q} \phi(0)| k^\mu_n \rangle \langle k^\mu_n | e^{-Q} \phi(0)| \Omega \rangle \theta(q_0)$$

Now there is no positivity requirement and we can cancel the $1/k^2$ behavior without having to give up unitarity. Since we still impose the reality of the momentum eigenvalues, the Hamiltonian of the theory must be $PT$ invariant rather than Hermitian. From the Lehmann representation we thus conclude that the Hamiltonian of theories such as the fourth-order Pais-Uhlenbeck oscillator cannot be Hermitian and must instead be $PT$ invariant, just as we had found directly.

**THE REMARKABLE MORAL OF THE STORY**

Consider ANY higher derivative theory in which the equation of motion is of the form $f(D)\phi = 0$ where $D = \partial_\mu \partial^\mu$ and $f(D) = \sum a_n D^n$. Also require that all momentum eigenvalues be real. In such a theory, at large $k^2$ the propagator will behave as $1/k^{2n}$. Hence there will be a contradiction with the Lehmann representation if we require the standard $\sum |n\rangle \langle n| = 1$. Rather, such theories must be $PT$ theories rather than standard Hermitian ones. Hence once we depart from second-order equations we are forced to $PT$-invariant, non-Hermitian Hamiltonians. $PT$-invariance is thus the general rule, and it is only a historical accident (second-order theories were encountered first) that it was not discovered earlier.
DYNAMICAL SYMMETRY BREAKING
AND THE COSMOLOGICAL CONSTANT PROBLEM

The zero-point energy and cosmological constant problems are two separate problems. With dynamical symmetry breaking they solve each other.

The zero-point energy problem already exists in a free field theory, and as such is separate from any cosmological constant term that might be induced by spontaneous symmetry breaking. The cosmological constant is associated with the minimum of the symmetry breaking potential while the zero-point energy is associated with the fluctuations about it. Moreover, the zero-point term and the cosmological constant term even transform differently under a general coordinate transformation, the former possessing a fluid velocity and being maximally 3-symmetric, with the latter possessing no fluid velocity and being maximally 4-symmetric. The zero-point fluctuation term is associated with a perfect matter fluid in which both $\rho_m$ and $p_m$ are positive, so that $T_{\mu\nu} = (\rho_m + p_m) U_{\mu} U_{\nu} + p_m g_{\mu\nu}$. While the cosmological constant term is associated with a perfect fluid in which $p = -\rho$, so that $T_{\mu\nu} = -\Lambda g_{\mu\nu}$.

When the symmetry is broken dynamically by fermion condensates, it is the fermionic zero-point fluctuations which cause the change in the vacuum in the first place, to thus actually produce the cosmological constant. In this case the zero-point and cosmological constant terms are not independent, and are related in a way which allows each one to cancel the other, so that both the zero-point and cosmological constant problems solve each other.
For a free quantum fermion field, in a mode of the form
\[ \psi(x) = \sum_{\pm s} \int \frac{d^4p}{(2\pi)^3} \left( \frac{m}{E_p} \right)^{1/2} [b(p, s)u(p, s)e^{-ip\cdot x} + d^\dagger(p, s)v(p, s)e^{ip\cdot x}] \] (238)
the vacuum expectation value of \( T_{\mu\nu} \sim \bar{\psi}\gamma_\mu \partial_\nu \psi \) is due to the non-vanishing of
\[ \langle \Omega|d(p, s)d^\dagger(p, s)|\Omega\rangle = \langle \Omega|(d(p, s)d^\dagger(p, s) + d^\dagger(p, s)d(p, s))|\Omega\rangle, \] (239)
to yield the zero-point fluctuation contribution
\[ \langle \Omega|T_{\mu\nu}|\Omega\rangle \sim -2 \int \frac{d^3p}{(2\pi)^3} \frac{p_\mu p_\nu}{E_p}, \quad \rho_m = \langle \Omega|T_{00}|\Omega\rangle \sim -2 \int \frac{d^3p}{(2\pi)^3} E_p, \]
\[ p_m = \langle \Omega|T_{xx}|\Omega\rangle = \langle \Omega|T_{yy}|\Omega\rangle = \langle \Omega|T_{zz}|\Omega\rangle \sim -2 \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{3E_p}. \] (240)
and a perfect fluid form \( T_{\mu\nu} = (\rho_m + p_m)U_\mu U_\nu + p_m g_{\mu\nu} \) where \( \rho_m \) and \( p_m \) have the SAME sign. That this is NOT a perfect fluid with \( p_m = -\rho_m \) is because the averaging is only 3-dimensional, i.e. for modes which obey the wave equation only the 3-momentum is averaged over. Moreover, the velocity of the fluid \( U_\mu \) is a timelike vector, and is allowed simply because a Minkowski metric distinguishes between timelike and spacelike.

In flat space, for \( T_{\mu\nu} \sim k_\mu k_\nu/E_k \) one can also construct the total energy and momentum as \( P_\mu = \int d^3x T_{0\mu} \), to find that
\[ P_0 \sim E_k, \quad P_i = 0, \] (241)
with there being no zero-point total momentum. However one can also evaluate

\[ \int d^3x T_{ij} \sim \int d^3x \frac{k_i k_j}{E} \sim \delta_{ij} \]  

(242)

to thus yield a zero-point quadrupole pressure tensor. Thus the absence of a zero-point total momentum does not mean the absence of a zero-point pressure; and while such a term plays no role in flat space it still couples to gravity in curved space.

To underscore the fact that the zero-point term is not the same as a cosmological constant term, we note that for a free massless field where \( p^\mu p_\mu = 0 \), the energy-momentum tensor is traceless and \( \rho_m = 3p_m \), something not possible for a cosmological constant term, since a traceless \( \Lambda g_{\mu\nu} \) would require \( \Lambda = 0 \).

A cosmological constant term is induced when the symmetry is broken and adds a term of the form of \( T_{\mu\nu} = -\Lambda g_{\mu\nu} \). It yields a contribution to the trace of the form \( T_{\mu\mu} = -4\Lambda \), with the zero-point energy and the cosmological constant terms thus being different. In the presence of both the energy density is given by \( T_{00} = \rho_m + \Lambda \).

When mass is generated dynamically in a four-Fermi theory, the mean field is produced by zero-point fluctuations. The cosmological constant is thus induced by zero-point fluctuations also and can thus be related to the zero-point energy. Since the trace of \( T_{\mu\nu} \) is zero in a conformal invariant theory, \( 3p_m - \rho_m - 4\Lambda = 0 \), and the zero-point and cosmological constant terms cancel each other identically. Neither can be bigger or smaller than the other.
FOUR-FERMI MEAN-FIELD THEORY IN TWO DIMENSIONS

In $D = 2$ the four-Fermi theory is defined via

$$L = i\bar{\psi}\gamma^\mu\partial_\mu\psi - (g/2)[\bar{\psi}\psi]^2, \quad i\gamma^\mu\partial_\mu\psi - g[\bar{\psi}\psi]\psi = 0,$$

(243)

$$T^{\mu\nu} = i\bar{\psi}\gamma^\mu\partial^\nu\psi - g^{\mu\nu}\frac{g}{2}[\bar{\psi}\psi]^2, \quad T^\mu = i\bar{\psi}\gamma^\mu\partial_\mu\psi - g[\bar{\psi}\psi]^2 = 0,$$

(244)

with the energy-momentum tensor being traceless. In the mean-field approximation one looks for self-consistent, translation invariant, states $|S\rangle$ in which

$$\langle S|\bar{\psi}\psi|S\rangle = m/g, \quad \langle S|[\bar{\psi}\psi - m/g]^2|S\rangle = 0, \quad i\gamma^\mu\partial_\mu\psi - m\psi = 0,$$

(245)

$$\langle S|T^{\mu\nu}|S\rangle = \langle S|i\bar{\psi}\gamma^\mu\partial^\nu\psi|S\rangle - g^{\mu\nu}\frac{m^2}{2g}, \quad \langle S|T^\mu|S\rangle = m\langle S|\bar{\psi}\psi|S\rangle - \frac{m^2}{g} = 0.$$

(246)

with the mean-field approximation preserving tracelessness.

In a plane wave solution

$$\psi(x) = \sum_{\pm s} \int d^2p/(2\pi)^{1/2}(m/E_p)^{1/2}[b(p, s)u(p, s)e^{-ip\cdot x} + d^\dagger(p, s)v(p, s)e^{ip\cdot x}],$$

(247)

we find that

$$\langle \Omega|i\bar{\psi}\gamma^\mu\partial^\nu\psi|\Omega\rangle = -2 \int d^2pp^\mu p^\nu/(2\pi E_p)$$

(248)

due to zero-point fluctuations since

$$\langle \Omega|bb^\dagger|\Omega\rangle = \langle \Omega|(bb^\dagger + b^\dagger b)|\Omega\rangle.$$

(249)
**CANCELLATION OF ZERO-POINT AND \( \Lambda \) IN TRACE OF \( T_{\mu\nu} \)\)**

Incoherently adding together modes with \( p^\mu = (E_p, p) \) and \( (E_p, -p) \) gives

\[
\sum \frac{p^\mu p^\nu}{E_p} = \begin{pmatrix} 0 & \frac{p^2}{2E_p} \\ \frac{-p^2}{2E_p} & 0 \end{pmatrix} = (\rho_m + p_m)U^\mu U^\nu + p_m g^{\mu\nu}
\]

(250)

On setting \( m^2/g = g[\langle S|\overline{\psi}\psi|S\rangle]^2 = \Lambda \), we can thus set

\[
\langle S|T^{\mu\nu}|S\rangle = (\rho_m + p_m)U^\mu U^\nu + p_m g^{\mu\nu} - g^{\mu\nu} \Lambda, \quad \langle S|T_{\mu}^{\mu}|S\rangle = p_m - \rho_m - 2\Lambda = 0. \quad (251)
\]

Thus \( 2\Lambda = p_m - \rho_m \), neither bigger nor smaller. The reason for this is that all of \( \Lambda, \rho_m \) and \( p_m \) are determined in one the same state \(|S\rangle\). For the case of a fundamental scalar field \( \phi(x) \), \( \Lambda = \lambda \phi^4 \) is determined by the vacuum (location of the minimum of the Higgs double-well potential), while \( \rho_m \) and \( p_m \) are determined by whichever matter field frequency modes are occupied. Hence one cannot relate \( \Lambda \) to \( \rho_m \) and \( p_m \) in standard cosmology with a fundamental Higgs field, to thus give rise to the cosmological constant problem. If however scalar field is a c-number Ginzburg-Landau condensate \( \langle S|\overline{\psi}\psi|S\rangle \) then can relate relate \( \Lambda, \rho_m \) and \( p_m \), and even have them cancel each other.

**Comparison test:** If Higgs is a c-number condensate then will **NOT** be produced in an accelerator such as the LHC, while if Higgs is fundamental then will be produced in an accelerator.
CANCELLATION OF ZERO-POINT AND \( \Lambda \) IN \( T_{\mu \nu} \) ITSELF

In four spacetime dimensions invariance under

\[
g_{\mu \nu}(x) \rightarrow e^{2\alpha(x)}g_{\mu \nu}(x),
\]
leads to a unique gravitational action

\[
I_W = -\alpha_g \int d^4 x (-g)^{1/2} C_{\lambda \mu \nu \kappa} C^{\lambda \mu \nu \kappa} = -2\alpha_g \int d^4 x (-g)^{1/2} \left[ R^{\mu \nu} R_{\mu \nu} - \frac{1}{3} (R^\alpha_\alpha)^2 \right]
\]
where \( \alpha_g \) is dimensionless and \( C_{\lambda \mu \nu \kappa} \) is the conformal Weyl tensor. The associated gravitational equations of motion are the \textbf{fourth-order derivative}:

\[
4\alpha_g [2C^{\mu \lambda \nu \kappa}_{\; ; \lambda ; \kappa} - C^{\mu \lambda \nu \kappa} R_{\lambda \kappa}] = T^{\mu \nu}
\]
and thus in cosmology where \( C_{\lambda \mu \nu \kappa} = 0 \), we obtain

\[
T^{\mu \nu} = 0
\]
Thus again get cancellation of zero-point and \( \Lambda \) terms in \( T_{\mu \nu} \) itself, but now need to include zero-point fluctuations in gravitational field as well. While \( \langle S | [2C^{\mu \lambda \nu \kappa}_{\; ; \lambda ; \kappa} - C^{\mu \lambda \nu \kappa} R_{\lambda \kappa}] | S \rangle \) vanishes in a classical cosmological background, quantum-mechanically there is a zero-point fluctuation contribution. Using the gravitational zero-point fluctuations to cancel the matter field zero-point fluctuations and \( \Lambda \) is only achievable in a renormalizable theory of gravity. Hence works in conformal gravity but not in standard gravity.
\[
I = \frac{1}{2} \int d^4 x \left[ \partial_\mu \partial_\nu \phi \partial^\mu \partial^\nu \phi - M^2 \partial_\mu \phi \partial^\mu \phi \right]
\]

\[\frac{1}{k^2(k^2 - M^2)} = \frac{1}{M^2} \left( \frac{1}{k^2 - M^2} - \frac{1}{k^2} \right)\]

\[
\Delta^\text{int}_F (x - y) = i \langle \Omega | T[\phi(x)\phi(y)] | \Omega \rangle
\]

\[
\sum_n |n\rangle \langle n| = 1
\]

\[
\rho(q^2) = (2\pi)^3 \sum_n \delta^4(k_{\mu}^n - q_{\mu}) |\langle \Omega | \phi(0) | k_{\mu}^n \rangle|^2 \theta(q_0)
\]

\[
\Delta^\text{int}_F (x - y) = \int_0^\infty dm^2 \rho(m^2) \Delta^\text{free}_{(F,2)}(x - y; m^2)
\]

\[
D(\bar{x}, \bar{x}', E) = \sum_n \frac{\psi_n(\bar{x}) \psi^*_n(\bar{x}')}{E - E_n}
\]

\[
\sum_n |n\rangle \langle n| - \sum_m |m\rangle \langle m| = 1
\]
12 THE PAIS-UHLENBECK OSCILLATOR

\[ \phi(\bar{x}, t) \sim z(t)e^{i\bar{k} \cdot \bar{x}}, \quad \omega_1 = (\bar{k}^2 + M^2)^{1/2}, \quad \omega_2 = |\bar{k}| \]  

(265)

\[ \frac{d^4 z}{dt^4} + (\omega_1^2 + \omega_2^2) \frac{d^2 z}{dt^2} + \omega_1^2 \omega_2^2 z = 0 \]  

(266)

\[ I_{PU} = \frac{\gamma}{2} \int dt[z^2 - (\omega_1^2 + \omega_2^2) \dot{z}^2 + \omega_1^2 \omega_2^2 z^2] \]  

(267)

\[ H_{PU} = \frac{p_x^2}{2\gamma} + p_z x + \frac{\gamma}{2} (\omega_1^2 + \omega_2^2) x^2 - \frac{\gamma}{2} \omega_1^2 \omega_2^2 z^2, \quad x = \dot{z} \]  

(268)

\[ [x, p_x] = i, \quad [z, p_z] = i \]  

(269)

\[ z = a_1 + a_1^\dagger + a_2 + a_2^\dagger, \quad p_z = i\gamma \omega_1 \omega_2^2 (a_1 - a_1^\dagger) + i\gamma \omega_1 \omega_2 (a_2 - a_2^\dagger), \]  

\[ x = -i\omega_1 (a_1 - a_1^\dagger) - i\omega_2 (a_2 - a_2^\dagger), \quad p_x = -\gamma \omega_1^2 (a_1 + a_1^\dagger) - \gamma \omega_2^2 (a_2 + a_2^\dagger) \]  

(270)

\[ H_{PU} = 2\gamma (\omega_1^2 - \omega_2^2) (\omega_2^2 a_1^\dagger a_1 - \omega_1^2 a_2^\dagger a_2) + (\omega_1 + \omega_2)/2 \]  

(271)

\[ [a_1, a_1^\dagger] = \frac{1}{2\gamma \omega_1 (\omega_1^2 - \omega_2^2)}, \quad [a_2, a_2^\dagger] = -\frac{1}{2\gamma \omega_2 (\omega_1^2 - \omega_2^2)} \]  

(272)

\[ a_1 |\Omega\rangle = a_2 |\Omega\rangle = 0, \quad H_{PU} |\Omega\rangle = \frac{1}{2}(\omega_1 + \omega_2) |\Omega\rangle, \quad \langle \Omega | a_2 a_2^\dagger |\Omega\rangle < 0, \]  

\[ a_1 |\hat{\Omega}\rangle = a_2 a_2^\dagger |\hat{\Omega}\rangle = 0, \quad H_{PU} |\hat{\Omega}\rangle = \frac{1}{2}(\omega_1 - \omega_2) |\hat{\Omega}\rangle, \quad \langle \hat{\Omega} | a_2 a_2^\dagger |\hat{\Omega}\rangle > 0 \]  

(273)
13  SOLUTION: HAMILTONIAN NOT HERMITIAN – BUT IT IS PT SYMMETRIC

\[ p_x = -i \frac{\partial}{\partial x}, \quad p_z = -i \frac{\partial}{\partial z} \]  \hspace{1cm} (274)

\[ \psi_0(z, x) = \exp \left[ \frac{\gamma}{2}(\omega_1 + \omega_2)\omega_2 z^2 + i\gamma\omega_1\omega_2 zx - \frac{\gamma}{2}(\omega_1 + \omega_2)x^2 \right] \]  \hspace{1cm} (275)

\[ [z, p_z] = i, \quad \left[ e^{i\theta} z, -\frac{i}{e^{i\theta}} \frac{\partial}{\partial z} \right] \psi(e^{i\theta} z) = i\psi(e^{i\theta} z) \]  \hspace{1cm} (276)

\[ y = e^{p_z z/2} e^{-p_z z/2} = -iz, \quad q = e^{p_z z/2} p_z e^{-p_z z/2} = ip_z \]  \hspace{1cm} (277)

\[ H = \frac{p_x^2}{2\gamma} - iq x + \frac{\gamma}{2} \left( \omega_1^2 + \omega_2^2 \right) x^2 + \frac{\gamma}{2} \omega_1^2 \omega_2^2 y^2, \quad p = p_x \]  \hspace{1cm} (278)

\[ [p, x] = i, \quad [q, y] = i \]  \hspace{1cm} (279)

\[ C^2 = 1, \quad [C, PT] = 0, \quad [C, H] = 0, \quad C = e^Q P \]  \hspace{1cm} (280)

\[ Q = \alpha[pq + \gamma^2 \omega_1^2 \omega_2^2 xy], \quad \alpha = \frac{1}{\gamma\omega_1\omega_2} \log \left( \frac{\omega_1 + \omega_2}{\omega_1 - \omega_2} \right) \]  \hspace{1cm} (281)

\[ \tilde{H} = e^{-Q/2} He^{Q/2} = \frac{p_x^2}{2\gamma} + \frac{q_x^2}{2\gamma\omega_1^2} + \frac{\gamma}{2} \omega_1^2 x^2 + \frac{\gamma}{2} \omega_1^2 \omega_2^2 y^2 \]  \hspace{1cm} (282)
\[ \tilde{H} = e^{-Q/2} H e^{Q/2} = \frac{p^2}{2\gamma} + \frac{q^2}{2\gamma\omega_1^2} + \frac{\gamma^2}{2} \omega_1^2 x^2 + \frac{\gamma^2}{2} \omega_1^2 \omega_2^2 y^2 \]  

(283)

\[ \tilde{H} |\tilde{n}\rangle = E_n |\tilde{n}\rangle, \quad H |n\rangle = E_n |n\rangle, \quad |n\rangle = e^{Q/2} |\tilde{n}\rangle \]  

(284)

\[ \langle \tilde{n}|\tilde{H} = E_n \langle \tilde{n}|, \quad \langle n| \equiv \langle \tilde{n}| e^{Q/2}, \quad \langle n|e^{-Q} H = \langle n|e^{-Q} E_n \]  

(285)

\[ \langle \tilde{n}|\tilde{m} = \delta_{m,n}, \quad \sum_n |\tilde{n}\rangle \langle \tilde{n}| = 1, \quad \tilde{H} = \sum_n |\tilde{n}\rangle E_n \langle \tilde{n}| \]  

(286)

\[ \langle n|e^{-Q}|m = \delta_{m,n}, \quad \sum_n |n\rangle \langle n|e^{-Q} = 1, \quad H = \sum_n |n\rangle E_n \langle n|e^{-Q} \]  

(287)

\[ \sum_n |n\rangle \langle n|PC = 1, \quad H = \sum_n |n\rangle E_n \langle n|PC, \quad C_n = \pm 1 \]  

(288)

\[ \langle x,y| e^{-iHt} |x',y'\rangle = \sum_n \psi_n(x,y) C_n e^{-iE_n t} \psi_n^c(x',y') \]  

(289)
\[ i \frac{d}{dt} |\alpha_S(t)\rangle = H |\alpha_S(t)\rangle, \quad -i \frac{d}{dt} \langle \alpha_S(t) | = \langle \alpha_S(t) | H^\dagger \] (290)

\[ |\alpha_S(t)\rangle = e^{-iHt} |\alpha_S(0)\rangle, \quad \langle \alpha_S(t) | = \langle \alpha_S(0) | e^{iH^\dagger t} \] (291)

\[ \langle \alpha_S(t) | \alpha_S(t) \rangle = \langle \alpha_S(0) | e^{iH^\dagger t} e^{-iHt} |\alpha_S(0)\rangle \neq \langle \alpha_S(0) | \alpha_S(0) \rangle \] (292)

\[ \langle \alpha_S(t) | A_S | \alpha_S(t) \rangle = \langle \alpha_S(0) | e^{iH^\dagger t} A_S e^{-iHt} |\alpha_S(0)\rangle \] (293)

\[ A_H(t) = e^{iH^\dagger t} A_S e^{-iHt} \] (294)

\[ i \frac{d}{dt} A_H(t) = A_H(t) H - H^\dagger A_H(t), \quad i \frac{d}{dt} A_H(t) = A_H(t) H - H A_H(t) \] (295)

\[ i \frac{d}{dt} |\alpha_S(t)\rangle = H |\alpha_S(t)\rangle, \quad -i \frac{d}{dt} \langle \hat{\alpha}_S(t) | = \langle \hat{\alpha}_S(t) | H \] (296)

\[ |\alpha_S(t)\rangle = e^{-iHt} |\alpha_S(0)\rangle, \quad \langle \hat{\alpha}_S(t) | = \langle \hat{\alpha}_S(0) | e^{iHt} \] (297)

\[ \langle \hat{\alpha}_S(t) | \alpha_S(t) \rangle = \langle \hat{\alpha}_S(0) | e^{iHt} e^{-iHt} |\alpha_S(0)\rangle = \langle \hat{\alpha}_S(0) | \alpha_S(0) \rangle \] (298)

\[ \langle \hat{\alpha}_S(t) | = \langle \alpha_S(t) | e^{-Q}, \quad H^\dagger = e^{-Q} H e^Q \] (299)
\[ \psi_0(x, y, t) = \exp \left[ -\frac{\gamma}{2} (\omega_1 + \omega_2)(x^2 + \omega_1 \omega_2 y^2) - \gamma \omega_1 \omega_2 y x \right] \exp(-i E_0 t), \quad E_0 = (\omega_1 + \omega_2)/2 \]  
\[ (300) \]

\[ \psi_1(x, y, t) = (x + \omega_2 y) \psi_0(x, y, t) e^{-i \omega_1 t}, \quad E_1 = E_0 + \omega_1 \]
\[ \psi_2(x, y, t) = (x + \omega_1 y) \psi_0(x, y, t) e^{-i \omega_2 t}, \quad E_2 = E_0 + \omega_2 \] 
\[ (301) \]

\[ \psi_3(x, y, t) = \left( (x + \omega_2 y)^2 - 1/2 \gamma \omega_1 \right) \psi_0(x, y, t) e^{-2i \omega_1 t}, \quad E_3 = E_0 + 2 \omega_1 \]
\[ \psi_4(x, y, t) = \left( (x + \omega_1 y)(x + \omega_2 y) - 1/\gamma (\omega_1 + \omega_2) \right) \psi_0(x, y, t) e^{-i (\omega_1 + \omega_2) t}, \quad E_4 = E_0 + \omega_1 + \omega_2 \]
\[ \psi_5(x, y, t) = \left( (x + \omega_1 y)^2 - 1/2 \gamma \omega_2 \right) \psi_0(x, y, t) e^{-2i \omega_2 t}, \quad E_5 = E_0 + 2 \omega_2 \] 
\[ (302) \]

\[ \hat{\psi}_0(x, y, t) = \exp \left[ -\gamma \omega^3 y^2 - \gamma \omega^2 y x - \gamma \omega x^2 - i \omega t \right], \quad \hat{E}_0 = \omega \] 
\[ (303) \]

\[ \hat{\psi}_1(x, y, t) = (x + \omega y) \hat{\psi}_0(x, y, t) e^{-i \omega t}, \quad \hat{E}_1 = \hat{E}_0 + \omega \] 
\[ (304) \]

\[ \hat{\psi}_2(x, y, t) = \left[ (x + \omega y)^2 - 1/2 \gamma \omega \right] \hat{\psi}_0(x, y, t) e^{-2i \omega t}, \quad \hat{E}_2 = \hat{E}_0 + 2 \omega \] 
\[ (305) \]
\[
H_{1P}(\epsilon) = \frac{1}{2\omega} \begin{pmatrix}
4\omega^2 + \epsilon^2 & 4\omega^2 - \epsilon^2 \\
\epsilon^2 & 4\omega^2 - \epsilon^2
\end{pmatrix}
\] (306)

\[
|2\omega + \epsilon\rangle \equiv \begin{pmatrix} 2\omega + \epsilon \\ \epsilon \end{pmatrix}, \quad |2\omega - \epsilon\rangle \equiv \begin{pmatrix} 2\omega - \epsilon \\ -\epsilon \end{pmatrix}
\] (307)

\[
S^{-1} \left( \frac{1}{2\omega} \right) \begin{pmatrix}
4\omega^2 + \epsilon^2 & 4\omega^2 - \epsilon^2 \\
\epsilon^2 & 4\omega^2 - \epsilon^2
\end{pmatrix} S = \begin{pmatrix} 2\omega + \epsilon & 0 \\ 0 & 2\omega - \epsilon \end{pmatrix}
\] (308)

\[
S = \frac{1}{2\epsilon\omega^{1/2}(2\omega + \epsilon)^{1/2}} \begin{pmatrix} 2\omega + \epsilon & -(4\omega^2 - \epsilon^2)\epsilon \\ \epsilon & (2\omega + \epsilon)\epsilon^2 \end{pmatrix},
\]

\[
S^{-1} = \frac{1}{2\epsilon\omega^{1/2}(2\omega + \epsilon)^{1/2}} \begin{pmatrix}(2\omega + \epsilon)\epsilon^2 & (4\omega^2 - \epsilon^2)\epsilon \\ -\epsilon & 2\omega + \epsilon \end{pmatrix}
\] (309)

\[
H_{1P}(\epsilon = 0) = 2\omega \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}
\] (310)

\[
\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} c + d \\ d \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}
\] (311)
\[ \omega_1 \equiv \omega + \epsilon, \quad \omega_2 \equiv \omega - \epsilon, \quad \epsilon \to 0 \] (312)

\[ \hat{\psi}_{1a}(x, y, t) = \lim_{\epsilon \to 0} \frac{\psi_2(x, y, t) - \psi_1(x, y, t)}{2\epsilon} \]
\[ = [(x + \omega y)it + y] \hat{\psi}_0(x, y, t)e^{-i\omega t} \] (313)

\[ \hat{\psi}_{2a}(x, y, t) = \lim_{\epsilon \to 0} \frac{\psi_5(x, y, t) - \psi_3(x, y, t)}{2\epsilon} \]
\[ = \left[ \left( (x + \omega y)^2 - \frac{1}{2\gamma \omega} \right) 2it + 2xy + 2\omega y^2 - \frac{1}{2\gamma \omega^2} \right] \hat{\psi}_0(x, y, t)e^{-2i\omega t} \] (314)

\[ \hat{\psi}_{2b}(x, y, t) = \lim_{\epsilon \to 0} \frac{2\psi_4(x, y, t) - \psi_3(x, y, t) - \psi_5(x, y, t)}{2\epsilon^2} \]
\[ = \left[ \left( (x + \omega y)^2 - \frac{1}{2\gamma \omega} \right) 2t^2 - \left( 2xy + 2\omega y^2 - \frac{1}{2\gamma \omega^2} \right) 2it \right. \]
\[ -2y^2 + \frac{1}{2\gamma \omega^3} \hat{\psi}_0(x, y, t)e^{-2i\omega t} \] (315)

\[ i \frac{\partial}{\partial t} \hat{\psi}(x, y, t) = \left( -\frac{1}{2\gamma} \frac{\partial^2}{\partial x^2} - x \frac{\partial}{\partial y} + \gamma \omega^2 x^2 + \frac{\gamma}{2} \omega^4 y^2 \right) \hat{\psi}(x, y, t) \] (316)

\[ i \frac{\partial}{\partial t} \int dxdy \hat{\psi}_B^c(x, y, t) \hat{\psi}_A(x, y, t) = -\int dxdy x \frac{\partial}{\partial y} \left[ \hat{\psi}_B^c(x, y, t) \hat{\psi}_A(x, y, t) \right] \] (317)

\[ i \frac{\partial}{\partial t} \int dxdy \hat{\psi}_B^c(x, y, t) \hat{\psi}_A(x, y, t) = 0 \] (318)
Conformal Supergravity in Twistor-String Theory

N. Berkovits and E. Witten


“The net effect is that the translation generator $D$ acts as

$$
\begin{pmatrix}
P & * \\
0 & P
\end{pmatrix}
$$

where $P$ would represent ordinary translations and the off-diagonal $*$ arises from $[D, \partial_I] \neq 0$.

This matrix is not diagonalizable. This clashes with our usual experience. We are accustomed to the idea that the translation generators are Hermitian operators and so can be diagonalized. However, conformal supergravity is not a unitary theory, and one symptom of this is that the translation generators are undiagonalizable.”

CONFORMAL GRAVITY CHALLENGES STRING THEORY – UNITARITY OF FOURTH ORDER DERIVATIVE THEORIES

Philip D. Mannheim, Presentation at ICHEP-08, July 2008

GHOST PROBLEM AND UNITARITY


PT QUANTUM MECHANICS


PAIS-UHLENBECK FOURTH ORDER OSCILLATOR


CONFORMAL GRAVITY AND THE COSMOLOGICAL CONSTANT PROBLEM


**TWO BIG PROBLEMS:**

**COSMOLOGICAL CONSTANT** and **QUANTUM GRAVITY**

**SHOULD HAVE A COMMON SOLUTION**

To solve them both at once need a theory of gravity with dimensionless coupling constants and no fundamental mass scales. Then theory is renormalizable and there is no cosmological constant at the level of the Lagrangian. To achieve impose invariance under local conformal transformations of the form

\[ g_{\mu\nu}(x) \rightarrow e^{2\alpha(x)} g_{\mu\nu}(x), \]  

(319)
as it leads to a unique gravitational action

\[ I_W = -\alpha_g \int d^4x (-g)^{1/2} C_{\lambda\mu\nu\kappa} C^{\lambda\mu\nu\kappa} \]

\[ = -2\alpha_g \int d^4x (-g)^{1/2} \left[ R^{\mu\nu} R_{\mu\nu} - \frac{1}{3} (R^\alpha_\alpha)^2 \right] \] (320)

where \( \alpha_g \) is dimensionless and \( C_{\lambda\mu\nu\kappa} \) is the conformal Weyl tensor. The associated gravitational equations of motion are the fourth-order derivative:

\[ 4\alpha_g [2C^{\mu\lambda\nu\kappa};\lambda;\kappa - C^{\mu\lambda\nu\kappa} R_{\lambda\kappa}] = T^{\mu\nu} \] (321)

In the theory \( G_N \) is induced dynamically (just like the weak interaction \( G_F \)), with this \( G_N \) only controlling local physics, and with the standard solar system predictions remaining intact. Because of early universe phase transitions an electroweak scale \( \Lambda \) is induced dynamically as the universe cools. However, at the same time an effective global cosmological \( G_{\text{eff}} \) is also induced, with this \( G_{\text{eff}} \) being found to naturally be altogether smaller than \( G_N \). In the theory the standard \( \Omega_\Lambda = 8\pi G_N \Lambda/3cH^2 \) is replaced by \( \bar{\Omega}_\Lambda = 8\pi G_{\text{eff}} \Lambda/3cH^2 \), and one can prove a bound

\[ 0 \leq \bar{\Omega}_\Lambda \leq 1, \quad -1 \leq q_0 \leq 0 \] (322)
no matter how big $\Lambda$ might be. The contribution of $\Lambda$ to cosmic evolution is thus naturally tamed without any fine-tuning. In conformal gravity then one does not need to quench $\Lambda$ at all. Rather, one quenches the amount by which $\Lambda$ gravitates.
Figure 11: The $q_0 = -0.37$ conformal gravity fit (upper curve) and the $\Omega_M(t_0) = 0.3$, $\Omega_\Lambda(t_0) = 0.7$ standard model fit (lower curve) to the $z < 1$ supernovae Hubble plot data.
Figure 12: Hubble plot expectations for $q_0 = -0.37$ (highest curve) and $q_0 = 0$ (middle curve) conformal gravity and for $\Omega_M(t_0) = 0.3$, $\Omega_\Lambda(t_0) = 0.7$ standard gravity (lowest curve).
WHY WE BELIEVE IN DARK MATTER AND DARK ENERGY
— AND DO WE HAVE TO?

Philip D. Mannheim

University of Connecticut

Colloquium in Montreal

November 2010
Gravity is attractive. The sun shines

ALL YOU NEED TO KNOW ABOUT NEWTONIAN GRAVITY

\[ \phi(r > r_0) = -\frac{MG}{r} \quad \Rightarrow \quad \text{no precession of planets} \quad \text{except there is (323)} \]

\[ \nabla^2 \phi = -G\rho, \quad M = \int_0^{r_0} d^3r \rho \quad \text{not tested or needed in } r < r_0 \]

\[ m \frac{d^2r}{dt^2} = -\frac{mMG}{r^2} \quad \text{for } r > r_0, \quad \vec{F} = 0 \quad \Rightarrow \quad \vec{v} = \text{constant} \]

ALL YOU NEED TO KNOW ABOUT SPECIAL RELATIVITY

All observers moving with constant velocity must agree on same physics

\[ x^\mu = (ct, x, y, z), \quad ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -c^2 t^2 (1 - v^2/c^2) \]

\[ \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

\[ m \frac{d^2x^\mu}{ds^2} = f^\mu. \]

Newton’s law of gravity does not obey special relativity

Observers are allowed to accelerate
ALL YOU NEED TO KNOW ABOUT GENERAL RELATIVITY

\[ ds^2 = g_{\mu\nu}dx^\mu dx^\nu, \quad g^{\mu\lambda}g_{\nu\lambda} = g^\lambda_\nu = \delta_\nu^\lambda, \quad g_{\mu\nu} = g_{\mu\nu}(x) \] (329)

\[ \Gamma^\mu_{\nu\sigma} = \frac{1}{2}g^{\mu\lambda} \left[ \partial_\nu g_{\lambda\sigma} + \partial_\sigma g_{\lambda\nu} - \partial_\lambda g_{\nu\sigma} \right] \] (330)

geodesic : \[ \frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\nu\sigma} \frac{dx^\nu}{ds} \frac{dx^\sigma}{ds} = 0, \quad m \left( \frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\nu\sigma} \frac{dx^\nu}{ds} \frac{dx^\sigma}{ds} \right) = f^\mu \] (331)

\[ R^\lambda_{\mu\nu\kappa} = \frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^\kappa} + \Gamma^\lambda_{\kappa\eta} \Gamma^\eta_{\mu\nu} - \frac{\partial \Gamma^\mu_{\lambda\kappa}}{\partial x^\nu} - \Gamma^\lambda_{\nu\kappa} \Gamma^\eta_{\mu\eta} \quad \text{spacetime is flat if and only if } R^\lambda_{\mu\nu\kappa} = 0. \] (332)

\[ R_{\mu\kappa} = g^\nu_\lambda R^\lambda_{\mu\nu\kappa}, \quad R^\mu_\mu = g^{\mu\kappa}R_{\mu\kappa} \] (333)

\[ I_{EH} = -\frac{1}{16\pi G} \int d^4x (-g)^{1/2} R^\alpha_\alpha \] (334)

\[ -\frac{1}{8\pi G} \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R^\alpha_\alpha \right) = T_{\mu\nu} \] (335)

\[ \nabla^2 \phi = \rho, \quad \phi = -\frac{\beta}{r}, \quad -g_{00} = \frac{1}{g_{rr}} = 1 - \frac{2MG}{r} \] (336)

Space is not flat – it is curved – gives gravitational bending of light and precession

\[ \nabla^4 \phi = \rho, \quad \phi = -\frac{\beta}{r} + \frac{\gamma r}{2} \quad \text{Second order Poisson equation not unique} \] (337)
ALL YOU NEED TO KNOW ABOUT QUANTUM MECHANICS

We live in Hilbert space, not coordinate space

\[ x^\mu p^\nu - p^\nu x^\mu = i\hbar \eta^{\mu\nu}, \quad \psi(x) = \langle x | \psi \rangle \]  

(338)

ALL YOU NEED TO KNOW ABOUT QUANTUM FIELD THEORY

\[ I = -\int d^4x \left[ \frac{1}{2} \partial_\mu S \partial^\mu S + \frac{1}{2} m^2 S^2 \right] \]  

(339)

\[ \partial_\mu \partial^\mu S - m^2 S^2 = 0, \quad S = \int \frac{d^3k}{(2\pi)^3 E_k} \left[ a(\vec{k}) e^{i\vec{k} \cdot \vec{x} - iE_k t} + a^\dagger(\vec{k}) e^{-i\vec{k} \cdot \vec{x} + iE_k t} \right], \quad E_k = (\vec{k}^2 + m^2)^{1/2} \]  

(340)

\[ T_{\mu\nu} = \partial_\mu S \partial_\nu S - \frac{1}{2} \eta_{\mu\nu} \left[ \partial_\alpha S \partial^\alpha S + m^2 S^2 \right] \]  

(341)

quantize: \[ [a(\vec{k}), a^\dagger(\vec{k}')] = \delta^3 (\vec{k} - \vec{k}'), \quad a(\vec{k}) |\Omega\rangle = 0 \]

\[ \langle \Omega | T_{00} | \Omega \rangle = \frac{K^4}{8\pi^2} + \frac{m^2 K^2}{8\pi^2} - \frac{m^4}{32\pi^2} \ln \left( \frac{4K^2}{m^2} \right) + \frac{m^4}{32\pi^2} \]  

(342)

Gives INFINITE zero-point energy density to vacuum of every quantized system

95
ALL YOU NEED TO KNOW ABOUT PHASE TRANSITIONS

Free energy is released in a phase transition

Expanding, cooling universe undergoes a phase transition at around $10^{12}\text{eV}=10^{16}$ degrees

Free energy released $\sim \sigma T^4 \sim 10^{60}$

Known as cosmological constant

Current temperature of the universe is 3 degrees, i.e. Energy $\sim O(1)$
SO WHAT HAPPENS IF YOU PUT ALL THESE THINGS TOGETHER?

A TOTAL DISASTER

*SOLUTION*

NO CLASSICAL CURVATURE

ALL CURVATURE DUE TO QUANTUM MECHANICS

WITHOUT CURVATURE – SPACETIME WOULD BE FLAT
IMPACT OF A GLOBAL QUADRATIC POTENTIAL ON GALACTIC ROTATION CURVES

Philip D. Mannheim

University of Connecticut

Presentation at Fort Lauderdale
December 2010
IMPACT OF A GLOBAL QUADRATIC POTENTIAL ON GALACTIC ROTATION CURVES

Philip D. Mannheim
University of Connecticut

Seminar in Mexico City
November 2010
PDM: Alternatives to dark matter and dark energy

PDM and J. G. O’Brien:
Impact of a global quadratic potential on galactic rotation curves

Fitting galactic rotation curves with conformal gravity and a global quadratic potential
arXiv:1011.3495 [astro-ph.CO], November 2010
ALL YOU NEED TO KNOW ABOUT EINSTEIN GRAVITY

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad g^{\mu\lambda} g_{\mu\nu} = g^{\lambda\nu} = \delta^\lambda_\nu, \quad g_{\mu\nu} = g_{\mu\nu}(x) \]  
(343)

\[ \Gamma^\mu_{\nu\sigma} = \frac{1}{2} g^{\mu\lambda} \left[ \partial_\nu g_{\lambda\sigma} + \partial_\sigma g_{\lambda\nu} - \partial_\lambda g_{\nu\sigma} \right] \]  
(344)

geodesic: \[ \frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\nu\sigma} \frac{dx^\nu}{ds} \frac{dx^\sigma}{ds} = 0, \quad m \left( \frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\nu\sigma} \frac{dx^\nu}{ds} \frac{dx^\sigma}{ds} \right) = f^\mu \quad \text{general coordinate vector} \]  
(345)

\[ R^\lambda_{\mu\nu\kappa} = \frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^\kappa} + \Gamma^\lambda_{\kappa\eta} \Gamma^\eta_{\mu\nu} - \frac{\partial \Gamma^\lambda_{\mu\kappa}}{\partial x^\nu} - \Gamma^\lambda_{\nu\eta} \Gamma^\eta_{\mu\kappa} \quad \text{spacetime is flat if and only if} \ R^\lambda_{\mu\nu\kappa} = 0. \]  
(346)

\[ R_{\mu\kappa} = g^\nu_{\lambda} R^\lambda_{\mu\nu\kappa}, \quad R^\mu_{\mu} = g^{\mu\kappa} R_{\mu\kappa} \quad \text{general coordinate scalar} \]  
(347)

\[ g = \det(g_{\mu\nu}), \quad I_{EH} = -\frac{1}{16\pi G} \int d^4x (-g)^{1/2} R^\alpha_\alpha, \quad -\frac{1}{8\pi G} \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R^\alpha_\alpha \right) = T_{\mu\nu} \]  
(348)

for weak gravity reduces to: \[ \nabla^2 \phi = \rho, \quad \phi = -\frac{\beta}{r}, \]  
(349)

has vacuum Schwarzschild solution with \( T_{\mu\nu} = 0, \ R_{\mu\nu} = 0 \): \[ -g_{00} = \frac{1}{g_{rr}} = 1 - \frac{2MG}{r} \]  
(350)

Space is not flat – it is curved – gives gravitational bending of light and planetary precession
Figure 13: Keplerian expectation for planetary orbital velocities
Figure 14: Face on view of a spiral galaxy
Figure 15: Edge on view of a spiral galaxy
Figure 16: UGC 2885 rotation velocities from HII data
Figure 17: NGC 3198 rotation velocities from HI data
ALL YOU NEED TO KNOW ABOUT CONFORMAL GRAVITY

\[ g_{\mu\nu} \rightarrow e^{2\alpha(x)} g_{\mu\nu}, \quad C_{\lambda\mu\nu\kappa} \rightarrow e^{2\alpha(x)} C_{\lambda\mu\nu\kappa} \]  (351)

\[ C_{\lambda\mu\nu\kappa} = R_{\lambda\mu\nu\kappa} + \frac{1}{6} R^\alpha_{\quad \alpha} [g_{\lambda\nu} g_{\mu\kappa} - g_{\lambda\kappa} g_{\mu\nu}] - \frac{1}{2} [g_{\lambda\nu} R_{\mu\kappa} - g_{\lambda\kappa} R_{\mu\nu} - g_{\mu\nu} R_{\lambda\kappa} + g_{\mu\kappa} R_{\lambda\nu}] \]  (352)

\[ I_W = -\alpha g \int d^4 x (-g)^{1/2} C_{\lambda\mu\nu\kappa} C^{\lambda\mu\nu\kappa} = -2\alpha g \int d^4 x (-g)^{1/2} \left[ R^{\mu\nu} R_{\mu\nu} - \frac{1}{3} (R^\alpha_{\quad \alpha})^2 \right] \]  (353)

\[ 4\alpha g W^{\mu\nu} = T^{\mu\nu} \]  (354)

\[ W^{\mu\nu} = \frac{1}{2} g^{\mu\nu} (R^\alpha_{\quad \alpha})^{;\beta} + R^{\mu\nu;\beta} - R^{\mu;\beta}_{\quad ;\nu} - R^{\nu;\beta}_{\quad ;\mu} - 2 R^{\mu\beta} R^\nu_{\beta} + \frac{1}{2} g^{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} - \frac{2}{3} g^{\mu\nu} (R^\alpha_{\quad \alpha})^{;\beta} \]

\[ + \frac{2}{3} (R^\alpha_{\quad \alpha})^{;\mu;\nu} + \frac{2}{3} R^\alpha_{\quad \alpha} R^{\mu\nu} - \frac{1}{6} g^{\mu\nu} (R^\alpha_{\quad \alpha})^2 = 2 C^{\mu\nu\kappa}_{\quad ;\lambda;\kappa} - C^{\mu\lambda\nu\kappa}_{\quad ;\lambda;\kappa} R_{\lambda\kappa} \]  (355)

Weyl action is UNIQUE

\[ R_{\mu\nu} = 0 \] is also a vacuum solution So generalizes Schwarzschild and not just Newton

Unlike Einstein cannot add on cosmological constant \[ \int d^4 x (-g)^{1/2} \Lambda \]

Thus if conformal symmetry unbroken \[ \Lambda = 0 \]
Mannheim and Kazanas:

\[ ds^2 = -B(r) dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega_2, \quad B(r > r_0) = 1 - \frac{2\beta}{r} + \gamma r - kr^2. \] (356)

\[ \frac{3}{B(r)}(W^0_0 - W^r_r) = \nabla^4 B = B'''' + \frac{4B'''}{r} = f(r) = \frac{3}{4\alpha g B(r)}(T^0_0 - T^r_r), \] (357)

\[ \gamma = -\frac{1}{2} \int_0^{r_0} dr' r'^2 f(r'), \quad 2\beta = \frac{1}{6} \int_0^{r_0} dr' r'^4 f(r'). \] (358)

\[ \nabla^4 \phi = \rho, \quad \phi = -\frac{\beta}{r} + \frac{\gamma r}{2} \quad \text{Second order Poisson equation not unique} \] (359)

\[ V^*(r) = -\frac{\beta^* c^2}{r} + \frac{\gamma^* c^2 r}{2} \] (360)

Typically for spiral galaxies \( \Sigma(R) = \Sigma_0 e^{-R/R_0} \)

\[ \frac{v^2_{\text{LOC}}}{R} = \frac{N^* \beta^* c^2 R}{2R_0^3} \left[ I_0 \left( \frac{R}{2R_0} \right) K_0 \left( \frac{R}{2R_0} \right) - I_1 \left( \frac{R}{2R_0} \right) K_1 \left( \frac{R}{2R_0} \right) \right] + \frac{N^* \gamma^* c^2 R}{2R_0} I_1 \left( \frac{R}{2R_0} \right) K_1 \left( \frac{R}{2R_0} \right) \] (361)

\[ \frac{v^2_{\text{LOC}}}{R} \to \frac{N^* \beta^* c^2}{R^2} \left( 1 + \frac{9R_0^2}{2R^2} \right) + \frac{N^* \gamma^* c^2}{2} \left( 1 - \frac{3R_0^2}{2R^2} - \frac{45R_0^4}{8R^4} \right) \to \frac{N^* \beta^* c^2}{R^2} + \frac{N^* \gamma^* c^2}{2}, \] (362)
Comparing second order and fourth order

\[ \phi(r) = -\frac{1}{r} \int_{0}^{r} dr' r'^2 g(r') - \int_{r}^{\infty} dr' r' g(r'), \tag{363} \]

\[ \frac{d\phi(r)}{dr} = \frac{1}{r^2} \int_{0}^{r} dr' r'^2 g(r'). \tag{364} \]

Newtonian Gravity is LOCAL

\[ \phi(r) = -\frac{r}{2} \int_{0}^{r} dr' r'^2 h(r') - \frac{1}{6r} \int_{0}^{r} dr' r'^4 h(r') - \frac{1}{2} \int_{r}^{\infty} dr' r'^3 h(r') - \frac{r^2}{6} \int_{r}^{\infty} dr' r' h(r'). \tag{365} \]

\[ \frac{d\phi(r)}{dr} = -\frac{1}{2} \int_{0}^{r} dr' r'^2 h(r') + \frac{1}{6r^2} \int_{0}^{r} dr' r'^4 h(r') - \frac{r}{3} \int_{r}^{\infty} dr' r' h(r'), \quad h(r) = \frac{f(r)c^2}{2} \tag{366} \]

Conformal Gravity is GLOBAL

So cannot ignore the rest of the universe

Rest of universe has homogeneous and inhomogeneous matter
For homogeneous matter have Roberston-Walker Hubble flow

\[ \rho = \frac{4r}{2(1 + \gamma_0 r - kr^2)^{1/2} + 2 + \gamma_0 r}, \quad \tau = \int dtR(t) \] (367)

\[-(1 + \gamma_0 r - kr^2)c^2 dt^2 + \frac{dr^2}{(1 + \gamma_0 r - kr^2)} + r^2 d\Omega_2 =
\frac{1}{R^2(\tau)} \frac{[1 - \rho^2(\gamma_0^2/16 + k/4)]^2}{[(1 - \gamma_0\rho/4)^2 + k\rho^2/4]^2} \left[ -c^2 d\tau^2 + \frac{R^2(\tau)}{[1 - \rho^2(\gamma_0^2/16 + k/4)]^2} (d\rho^2 + \rho^2 d\Omega_2) \right]. \] (368)

\[ ds^2 = e^{2\alpha(\tau, \rho)} \left[ -c^2 d\tau^2 + \frac{R^2(\tau)}{[1 + K\rho^2/4]^2} (d\rho^2 + \rho^2 d\Omega_2) \right], \quad K = -\frac{\gamma_0^2}{4} - k \] (369)

But cannot get two parameters from one — so drop \( k \)

\[ \rho = \frac{4r}{2(1 + \gamma_0 r)^{1/2} + 2 + \gamma_0 r}, \quad \tau = \int dtR(t) \] (370)

\[-(1 + \gamma_0 r)c^2 dt^2 + \frac{dr^2}{(1 + \gamma_0 r)} + r^2 d\Omega_2 =
\frac{1}{R^2(\tau)} \left( \frac{1 + \gamma_0\rho/4}{1 - \gamma_0\rho/4} \right)^2 \left[ -c^2 d\tau^2 + \frac{R^2(\tau)}{[1 - \gamma_0^2\rho^2/16]^2} (d\rho^2 + \rho^2 d\Omega_2) \right], \quad K = -\frac{\gamma_0^2}{4} \] (371)

In rest frame a topologically open RW geometry acts like a universal linear potential
Figure 18: The $q_0 = -0.37$ conformal gravity fit (upper curve) and the $\Omega_M(t_0) = 0.3$, $\Omega_\Lambda(t_0) = 0.7$ standard model fit (lower curve) to the $z < 1$ supernovae Hubble plot data.
Figure 19: Hubble plot expectations for $q_0 = -0.37$ (highest curve) and $q_0 = 0$ (middle curve) conformal gravity and for $\Omega_M(t_0) = 0.3, \Omega_\Lambda(t_0) = 0.7$ standard gravity (lowest curve).
\[
\frac{v_{\text{TOT}}^2}{R} = \frac{v_{\text{LOC}}^2}{R} + \frac{\gamma_0 c^2}{2}.
\]

\[
\frac{v_{\text{TOT}}^2}{R} \to \frac{N^* \beta^* c^2}{R^2} + \frac{N^* \gamma^* c^2}{2} + \frac{\gamma_0 c^2}{2}.
\]

\(N^*\) = visible mass in solar mass units, \(\beta^* = 1.48 \times 10^5\) cm, \(\gamma^* = 5.42 \times 10^{-41}\) cm\(^{-1}\), \(\gamma_0 = 3.06 \times 10^{-30}\) cm\(^{-1}\).

Table 1: Characteristics of the eleven galaxy sample

<table>
<thead>
<tr>
<th>Galaxy</th>
<th>Distance (Mpc)</th>
<th>Luminosity ((10^9 L_{\odot}))</th>
<th>(R_0) (kpc)</th>
<th>(\frac{(v^2/c^2 R)_{\text{last}}}{10^{-30} \text{cm}^{-1}})</th>
<th>(\frac{(M/L)}{(M_{\odot} L_{\odot}^{-1})})</th>
</tr>
</thead>
<tbody>
<tr>
<td>DDO 154</td>
<td>3.80</td>
<td>0.05</td>
<td>0.48</td>
<td>1.51</td>
<td>0.71</td>
</tr>
<tr>
<td>DDO 170</td>
<td>12.01</td>
<td>0.16</td>
<td>1.28</td>
<td>1.63</td>
<td>5.36</td>
</tr>
<tr>
<td>NGC 1560</td>
<td>3.00</td>
<td>0.35</td>
<td>1.30</td>
<td>2.70</td>
<td>2.01</td>
</tr>
<tr>
<td>NGC 3109</td>
<td>1.70</td>
<td>0.81</td>
<td>1.55</td>
<td>1.98</td>
<td>0.01</td>
</tr>
<tr>
<td>UGC 2259</td>
<td>9.80</td>
<td>1.02</td>
<td>1.33</td>
<td>3.85</td>
<td>3.62</td>
</tr>
<tr>
<td>NGC 6503</td>
<td>5.94</td>
<td>4.80</td>
<td>1.73</td>
<td>2.14</td>
<td>3.00</td>
</tr>
<tr>
<td>NGC 2403</td>
<td>3.25</td>
<td>7.90</td>
<td>2.05</td>
<td>3.31</td>
<td>1.76</td>
</tr>
<tr>
<td>NGC 3198</td>
<td>9.36</td>
<td>9.00</td>
<td>2.72</td>
<td>2.67</td>
<td>4.78</td>
</tr>
<tr>
<td>NGC 2903</td>
<td>6.40</td>
<td>15.30</td>
<td>2.02</td>
<td>4.86</td>
<td>3.15</td>
</tr>
<tr>
<td>NGC 7331</td>
<td>14.90</td>
<td>54.00</td>
<td>4.48</td>
<td>5.51</td>
<td>3.03</td>
</tr>
<tr>
<td>NGC 2841</td>
<td>9.50</td>
<td>20.50</td>
<td>2.39</td>
<td>7.25</td>
<td>8.26</td>
</tr>
</tbody>
</table>
Figure 20: Some typical measured galactic rotation velocities
But also inhomogeneous matter – clusters and superclusters of galaxies

\[
\frac{v_{\text{TOT}}^2}{R} = \frac{v_{\text{LOC}}^2}{R} + \frac{\gamma_0 c^2}{2} - \kappa c^2 R, \tag{375}
\]

\[
\frac{v_{\text{TOT}}^2}{R} \rightarrow \frac{N^* \beta^* c^2}{R^2} + \frac{N^* \gamma^* c^2}{2} + \frac{\gamma_0 c^2}{2} - \kappa c^2 R. \tag{376}
\]

\[
\kappa = 9.54 \times 10^{-54}\text{cm}^{-2}. \tag{377}
\]
Table 1: Properties of the 20 Large Galaxy Sample

<table>
<thead>
<tr>
<th>Galaxy</th>
<th>Type</th>
<th>Distance (Mpc)</th>
<th>$L_B$ ($10^{10} L_\odot$)</th>
<th>$R_0$ (kpc)</th>
<th>$R_{last}$ (kpc)</th>
<th>$M_{HI}$ ($10^{10} M_\odot$)</th>
<th>$M_{disk}$ ($10^{10} M_\odot$)</th>
<th>$(M/L)_{stars}$</th>
<th>$(v^2/c^2 R)_{last}$ ($10^{-30} cm^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>NGC 3726</td>
<td>HSB</td>
<td>17.4</td>
<td>3.34</td>
<td>3.2</td>
<td>31.5</td>
<td>0.60</td>
<td>3.82</td>
<td>1.15</td>
<td>3.19</td>
</tr>
<tr>
<td>NGC 3769</td>
<td>HSB</td>
<td>15.5</td>
<td>0.68</td>
<td>1.5</td>
<td>32.2</td>
<td>0.41</td>
<td>1.36</td>
<td>1.99</td>
<td>1.43</td>
</tr>
<tr>
<td>NGC 4013</td>
<td>HSB</td>
<td>18.6</td>
<td>2.09</td>
<td>2.1</td>
<td>33.1</td>
<td>0.32</td>
<td>5.58</td>
<td>2.67</td>
<td>3.14</td>
</tr>
<tr>
<td>NGC 3521</td>
<td>HSB</td>
<td>12.2</td>
<td>4.77</td>
<td>3.3</td>
<td>35.3</td>
<td>1.03</td>
<td>9.25</td>
<td>1.94</td>
<td>4.21</td>
</tr>
<tr>
<td>NGC 2683</td>
<td>HSB</td>
<td>10.2</td>
<td>1.88</td>
<td>2.4</td>
<td>36.0</td>
<td>0.15</td>
<td>6.03</td>
<td>3.20</td>
<td>2.28</td>
</tr>
<tr>
<td>UGC 1230</td>
<td>LSB</td>
<td>54.1</td>
<td>0.37</td>
<td>4.7</td>
<td>37.1</td>
<td>0.65</td>
<td>0.67</td>
<td>1.82</td>
<td>0.97</td>
</tr>
<tr>
<td>NGC 3198</td>
<td>HSB</td>
<td>14.1</td>
<td>3.24</td>
<td>4.0</td>
<td>38.6</td>
<td>1.06</td>
<td>3.64</td>
<td>1.12</td>
<td>2.09</td>
</tr>
<tr>
<td>NGC 5371</td>
<td>HSB</td>
<td>35.3</td>
<td>7.59</td>
<td>4.4</td>
<td>41.0</td>
<td>0.89</td>
<td>8.52</td>
<td>1.44</td>
<td>3.98</td>
</tr>
<tr>
<td>NGC 2998</td>
<td>HSB</td>
<td>59.3</td>
<td>5.19</td>
<td>4.8</td>
<td>41.1</td>
<td>1.78</td>
<td>7.16</td>
<td>1.75</td>
<td>3.43</td>
</tr>
<tr>
<td>NGC 5055</td>
<td>HSB</td>
<td>9.2</td>
<td>3.62</td>
<td>2.9</td>
<td>44.4</td>
<td>0.76</td>
<td>6.04</td>
<td>1.87</td>
<td>2.36</td>
</tr>
<tr>
<td>NGC 5033</td>
<td>HSB</td>
<td>15.3</td>
<td>3.06</td>
<td>7.5</td>
<td>45.6</td>
<td>1.07</td>
<td>0.27</td>
<td>3.28</td>
<td>3.16</td>
</tr>
<tr>
<td>NGC 0801</td>
<td>HSB</td>
<td>63.0</td>
<td>4.75</td>
<td>9.5</td>
<td>46.7</td>
<td>1.39</td>
<td>6.93</td>
<td>2.37</td>
<td>3.59</td>
</tr>
<tr>
<td>NGC 5907</td>
<td>HSB</td>
<td>16.5</td>
<td>5.40</td>
<td>5.5</td>
<td>48.0</td>
<td>1.90</td>
<td>2.49</td>
<td>1.89</td>
<td>3.44</td>
</tr>
<tr>
<td>NGC 3992</td>
<td>HSB</td>
<td>25.6</td>
<td>8.46</td>
<td>5.7</td>
<td>49.6</td>
<td>1.94</td>
<td>13.94</td>
<td>1.65</td>
<td>4.08</td>
</tr>
<tr>
<td>NGC 2841</td>
<td>HSB</td>
<td>14.1</td>
<td>4.74</td>
<td>3.5</td>
<td>51.6</td>
<td>0.86</td>
<td>19.55</td>
<td>4.12</td>
<td>5.83</td>
</tr>
<tr>
<td>UGC 0128</td>
<td>LSB</td>
<td>64.6</td>
<td>0.60</td>
<td>6.9</td>
<td>54.8</td>
<td>0.73</td>
<td>2.75</td>
<td>4.60</td>
<td>1.03</td>
</tr>
<tr>
<td>NGC 5533</td>
<td>HSB</td>
<td>42.0</td>
<td>3.17</td>
<td>7.4</td>
<td>56.0</td>
<td>1.39</td>
<td>2.00</td>
<td>4.14</td>
<td>3.31</td>
</tr>
<tr>
<td>NGC 6674</td>
<td>HSB</td>
<td>42.0</td>
<td>4.94</td>
<td>7.1</td>
<td>59.1</td>
<td>2.18</td>
<td>2.00</td>
<td>2.52</td>
<td>3.57</td>
</tr>
<tr>
<td>UGC 6614</td>
<td>LSB</td>
<td>86.2</td>
<td>2.11</td>
<td>8.2</td>
<td>62.7</td>
<td>2.07</td>
<td>9.70</td>
<td>4.60</td>
<td>2.39</td>
</tr>
<tr>
<td>UGC 2885</td>
<td>HSB</td>
<td>80.4</td>
<td>23.96</td>
<td>13.3</td>
<td>74.1</td>
<td>3.98</td>
<td>8.47</td>
<td>0.72</td>
<td>4.31</td>
</tr>
</tbody>
</table>
Fitting to the rotational velocities (in km sec$^{-1}$) of the 20 large galaxy sample. No dark matter is assumed.
Table 2: Properties of the THINGS 18 Galaxy Sample

<table>
<thead>
<tr>
<th>Galaxy</th>
<th>Type</th>
<th>Distance (Mpc)</th>
<th>$L_B$ ($10^{10} L_\odot$)</th>
<th>$R_0$ (kpc)</th>
<th>$R_{last}$ (kpc)</th>
<th>$M_{HI}$ ($10^{10} M_\odot$)</th>
<th>$M_{disk}$ ($10^{10} M_\odot$)</th>
<th>($M/L$)$<em>{stars}$ ($M</em>\odot/L_\odot$)</th>
<th>($v^2/c^2 R$)$_{last}$ ($10^{-30} cm^{-1}$)</th>
<th>$v$</th>
</tr>
</thead>
<tbody>
<tr>
<td>DDO 0154</td>
<td>LSB</td>
<td>4.2</td>
<td>0.007</td>
<td>0.8</td>
<td>8.1</td>
<td>0.03</td>
<td>0.003</td>
<td>0.45</td>
<td>11.2</td>
<td>11.2</td>
</tr>
<tr>
<td>IC 2574</td>
<td>LSB</td>
<td>4.5</td>
<td>0.345</td>
<td>4.2</td>
<td>13.1</td>
<td>0.19</td>
<td>0.098</td>
<td>0.28</td>
<td>1.69</td>
<td>1.69</td>
</tr>
<tr>
<td>NGC 0925</td>
<td>LSB</td>
<td>8.7</td>
<td>1.444</td>
<td>3.9</td>
<td>12.4</td>
<td>0.41</td>
<td>1.372</td>
<td>0.95</td>
<td>4.17</td>
<td>4.17</td>
</tr>
<tr>
<td>NGC 2403</td>
<td>HSB</td>
<td>4.3</td>
<td>1.647</td>
<td>2.7</td>
<td>23.9</td>
<td>0.46</td>
<td>2.370</td>
<td>1.44</td>
<td>2.89</td>
<td>2.89</td>
</tr>
<tr>
<td>NGC 2841</td>
<td>HSB</td>
<td>14.1</td>
<td>4.742</td>
<td>3.5</td>
<td>51.6</td>
<td>0.86</td>
<td>19.552</td>
<td>4.12</td>
<td>5.83</td>
<td>5.83</td>
</tr>
<tr>
<td>NGC 2903</td>
<td>HSB</td>
<td>9.4</td>
<td>4.088</td>
<td>3.0</td>
<td>30.9</td>
<td>0.49</td>
<td>7.155</td>
<td>1.75</td>
<td>3.75</td>
<td>3.75</td>
</tr>
<tr>
<td>NGC 2976</td>
<td>LSB</td>
<td>3.6</td>
<td>0.201</td>
<td>1.2</td>
<td>2.6</td>
<td>0.01</td>
<td>0.322</td>
<td>1.60</td>
<td>10.43</td>
<td>10.43</td>
</tr>
<tr>
<td>NGC 3031</td>
<td>HSB</td>
<td>3.7</td>
<td>3.187</td>
<td>2.6</td>
<td>15.0</td>
<td>0.38</td>
<td>8.662</td>
<td>2.72</td>
<td>9.31</td>
<td>9.31</td>
</tr>
<tr>
<td>NGC 3198</td>
<td>HSB</td>
<td>14.1</td>
<td>3.241</td>
<td>4.0</td>
<td>38.6</td>
<td>1.06</td>
<td>3.644</td>
<td>1.12</td>
<td>2.09</td>
<td>2.09</td>
</tr>
<tr>
<td>NGC 3521</td>
<td>HSB</td>
<td>12.2</td>
<td>4.769</td>
<td>3.3</td>
<td>35.3</td>
<td>1.03</td>
<td>9.245</td>
<td>1.94</td>
<td>4.21</td>
<td>4.21</td>
</tr>
<tr>
<td>NGC 3621</td>
<td>HSB</td>
<td>7.4</td>
<td>2.048</td>
<td>2.9</td>
<td>28.7</td>
<td>0.89</td>
<td>2.891</td>
<td>1.41</td>
<td>3.18</td>
<td>3.18</td>
</tr>
<tr>
<td>NGC 3627</td>
<td>HSB</td>
<td>10.2</td>
<td>3.700</td>
<td>3.1</td>
<td>8.2</td>
<td>0.10</td>
<td>6.622</td>
<td>1.79</td>
<td>15.64</td>
<td>15.64</td>
</tr>
<tr>
<td>NGC 4736</td>
<td>HSB</td>
<td>5.0</td>
<td>1.460</td>
<td>2.1</td>
<td>10.3</td>
<td>0.05</td>
<td>1.630</td>
<td>1.60</td>
<td>4.66</td>
<td>4.66</td>
</tr>
<tr>
<td>NGC 4826</td>
<td>HSB</td>
<td>5.4</td>
<td>1.441</td>
<td>2.6</td>
<td>15.8</td>
<td>0.03</td>
<td>3.640</td>
<td>2.53</td>
<td>5.46</td>
<td>5.46</td>
</tr>
<tr>
<td>NGC 5055</td>
<td>HSB</td>
<td>9.2</td>
<td>3.622</td>
<td>2.9</td>
<td>44.4</td>
<td>0.76</td>
<td>6.035</td>
<td>1.87</td>
<td>2.36</td>
<td>2.36</td>
</tr>
<tr>
<td>NGC 6946</td>
<td>HSB</td>
<td>6.9</td>
<td>3.732</td>
<td>2.9</td>
<td>22.4</td>
<td>0.57</td>
<td>6.272</td>
<td>1.68</td>
<td>6.39</td>
<td>6.39</td>
</tr>
<tr>
<td>NGC 7331</td>
<td>HSB</td>
<td>14.2</td>
<td>6.773</td>
<td>3.2</td>
<td>24.4</td>
<td>0.85</td>
<td>12.086</td>
<td>1.78</td>
<td>9.61</td>
<td>9.61</td>
</tr>
<tr>
<td>NGC 7793</td>
<td>HSB</td>
<td>5.2</td>
<td>0.910</td>
<td>1.7</td>
<td>10.3</td>
<td>0.16</td>
<td>0.793</td>
<td>0.87</td>
<td>3.61</td>
<td>3.61</td>
</tr>
</tbody>
</table>
Table 3: Properties of the Ursa Major 30 Galaxy Sample

<table>
<thead>
<tr>
<th>Galaxy</th>
<th>Type</th>
<th>Distance (Mpc)</th>
<th>$L_B$ ($10^{10}L_\odot$)</th>
<th>$R_0$ (kpc)</th>
<th>$R_{last}$ (kpc)</th>
<th>$M_{HI}$ ($10^{10}M_\odot$)</th>
<th>$M_{disk}$ ($10^{10}M_\odot$)</th>
<th>$(M/L)<em>{stars}$ ($M</em>\odot/L_\odot$)</th>
<th>$(v^2/c^2R)_{last}$ ($10^{-30}$ cm$^{-1}$)</th>
<th>$v$</th>
</tr>
</thead>
<tbody>
<tr>
<td>NGC 3726</td>
<td>HSB</td>
<td>17.4</td>
<td>3.340</td>
<td>3.2</td>
<td>31.5</td>
<td>0.60</td>
<td>3.82</td>
<td>1.15</td>
<td>3.19</td>
<td></td>
</tr>
<tr>
<td>NGC 3769</td>
<td>HSB</td>
<td>15.5</td>
<td>0.684</td>
<td>1.5</td>
<td>32.2</td>
<td>0.41</td>
<td>1.36</td>
<td>1.99</td>
<td>1.43</td>
<td></td>
</tr>
<tr>
<td>NGC 3877</td>
<td>HSB</td>
<td>15.5</td>
<td>1.948</td>
<td>2.4</td>
<td>9.8</td>
<td>0.11</td>
<td>3.44</td>
<td>1.76</td>
<td>10.51</td>
<td></td>
</tr>
<tr>
<td>NGC 3893</td>
<td>HSB</td>
<td>18.1</td>
<td>2.928</td>
<td>2.4</td>
<td>20.5</td>
<td>0.59</td>
<td>5.00</td>
<td>1.71</td>
<td>3.85</td>
<td></td>
</tr>
<tr>
<td>NGC 3917</td>
<td>LSB</td>
<td>16.9</td>
<td>1.334</td>
<td>2.8</td>
<td>13.9</td>
<td>0.17</td>
<td>2.23</td>
<td>1.67</td>
<td>4.85</td>
<td></td>
</tr>
<tr>
<td>NGC 3949</td>
<td>HSB</td>
<td>18.4</td>
<td>2.327</td>
<td>1.7</td>
<td>7.2</td>
<td>0.35</td>
<td>2.37</td>
<td>1.02</td>
<td>14.23</td>
<td></td>
</tr>
<tr>
<td>NGC 3953</td>
<td>HSB</td>
<td>18.7</td>
<td>4.236</td>
<td>3.9</td>
<td>16.3</td>
<td>0.31</td>
<td>9.79</td>
<td>2.31</td>
<td>10.20</td>
<td></td>
</tr>
<tr>
<td>NGC 3972</td>
<td>HSB</td>
<td>18.6</td>
<td>0.978</td>
<td>2.0</td>
<td>9.0</td>
<td>0.13</td>
<td>1.49</td>
<td>1.53</td>
<td>7.18</td>
<td></td>
</tr>
<tr>
<td>NGC 3992</td>
<td>HSB</td>
<td>25.6</td>
<td>8.456</td>
<td>5.7</td>
<td>49.6</td>
<td>1.94</td>
<td>13.94</td>
<td>1.65</td>
<td>4.08</td>
<td></td>
</tr>
<tr>
<td>NGC 4010</td>
<td>LSB</td>
<td>18.4</td>
<td>0.883</td>
<td>3.4</td>
<td>10.6</td>
<td>0.29</td>
<td>2.03</td>
<td>2.30</td>
<td>5.03</td>
<td></td>
</tr>
<tr>
<td>NGC 4013</td>
<td>HSB</td>
<td>18.6</td>
<td>2.088</td>
<td>2.1</td>
<td>33.1</td>
<td>0.32</td>
<td>5.58</td>
<td>2.67</td>
<td>3.14</td>
<td></td>
</tr>
<tr>
<td>NGC 4051</td>
<td>HSB</td>
<td>14.6</td>
<td>2.281</td>
<td>2.3</td>
<td>9.9</td>
<td>0.18</td>
<td>3.17</td>
<td>1.39</td>
<td>8.52</td>
<td></td>
</tr>
<tr>
<td>NGC 4085</td>
<td>HSB</td>
<td>19.0</td>
<td>1.212</td>
<td>1.6</td>
<td>6.5</td>
<td>0.15</td>
<td>1.34</td>
<td>1.11</td>
<td>10.21</td>
<td></td>
</tr>
<tr>
<td>NGC 4088</td>
<td>HSB</td>
<td>15.8</td>
<td>2.957</td>
<td>2.8</td>
<td>18.8</td>
<td>0.64</td>
<td>4.67</td>
<td>1.58</td>
<td>5.79</td>
<td></td>
</tr>
<tr>
<td>NGC 4100</td>
<td>HSB</td>
<td>21.4</td>
<td>3.388</td>
<td>2.9</td>
<td>27.1</td>
<td>0.44</td>
<td>5.74</td>
<td>1.69</td>
<td>3.35</td>
<td></td>
</tr>
<tr>
<td>NGC 4138</td>
<td>LSB</td>
<td>15.6</td>
<td>0.827</td>
<td>1.2</td>
<td>16.1</td>
<td>0.11</td>
<td>2.97</td>
<td>3.59</td>
<td>5.04</td>
<td></td>
</tr>
<tr>
<td>NGC 4157</td>
<td>HSB</td>
<td>18.7</td>
<td>2.901</td>
<td>2.6</td>
<td>30.9</td>
<td>0.88</td>
<td>5.83</td>
<td>2.01</td>
<td>3.99</td>
<td></td>
</tr>
<tr>
<td>NGC 4183</td>
<td>HSB</td>
<td>16.7</td>
<td>1.042</td>
<td>2.9</td>
<td>19.5</td>
<td>0.30</td>
<td>1.43</td>
<td>1.38</td>
<td>2.36</td>
<td></td>
</tr>
<tr>
<td>NGC 4217</td>
<td>HSB</td>
<td>19.6</td>
<td>3.031</td>
<td>3.1</td>
<td>18.2</td>
<td>0.30</td>
<td>5.53</td>
<td>1.83</td>
<td>6.28</td>
<td></td>
</tr>
<tr>
<td>NGC 4389</td>
<td>HSB</td>
<td>15.5</td>
<td>0.610</td>
<td>1.2</td>
<td>4.6</td>
<td>0.04</td>
<td>0.42</td>
<td>0.68</td>
<td>9.49</td>
<td></td>
</tr>
<tr>
<td>UGC 6399</td>
<td>LSB</td>
<td>18.7</td>
<td>0.291</td>
<td>2.4</td>
<td>8.1</td>
<td>0.07</td>
<td>0.59</td>
<td>2.04</td>
<td>3.42</td>
<td></td>
</tr>
<tr>
<td>UGC 6446</td>
<td>LSB</td>
<td>15.9</td>
<td>0.263</td>
<td>1.9</td>
<td>13.6</td>
<td>0.24</td>
<td>0.36</td>
<td>1.36</td>
<td>1.70</td>
<td></td>
</tr>
<tr>
<td>UGC 6667</td>
<td>LSB</td>
<td>19.8</td>
<td>0.422</td>
<td>3.1</td>
<td>8.6</td>
<td>0.10</td>
<td>0.71</td>
<td>1.67</td>
<td>3.09</td>
<td></td>
</tr>
<tr>
<td>UGC 6818</td>
<td>LSB</td>
<td>21.7</td>
<td>0.352</td>
<td>2.1</td>
<td>8.4</td>
<td>0.16</td>
<td>0.11</td>
<td>0.33</td>
<td>2.35</td>
<td></td>
</tr>
<tr>
<td>UGC 6917</td>
<td>LSB</td>
<td>18.9</td>
<td>0.563</td>
<td>2.9</td>
<td>10.9</td>
<td>0.22</td>
<td>1.24</td>
<td>2.20</td>
<td>4.05</td>
<td></td>
</tr>
<tr>
<td>UGC 6923</td>
<td>LSB</td>
<td>18.0</td>
<td>0.297</td>
<td>1.5</td>
<td>5.3</td>
<td>0.08</td>
<td>0.35</td>
<td>1.18</td>
<td>4.43</td>
<td></td>
</tr>
<tr>
<td>UGC 6930</td>
<td>LSB</td>
<td>17.0</td>
<td>0.601</td>
<td>2.2</td>
<td>15.7</td>
<td>0.29</td>
<td>1.02</td>
<td>1.69</td>
<td>2.68</td>
<td></td>
</tr>
<tr>
<td>UGC 6973</td>
<td>LSB</td>
<td>25.3</td>
<td>1.647</td>
<td>2.2</td>
<td>11.0</td>
<td>0.35</td>
<td>3.99</td>
<td>2.42</td>
<td>10.58</td>
<td></td>
</tr>
<tr>
<td>UGC 6983</td>
<td>LSB</td>
<td>20.2</td>
<td>0.577</td>
<td>2.9</td>
<td>17.6</td>
<td>0.37</td>
<td>1.28</td>
<td>2.22</td>
<td>2.43</td>
<td></td>
</tr>
<tr>
<td>UGC 7089</td>
<td>LSB</td>
<td>13.9</td>
<td>0.352</td>
<td>2.3</td>
<td>7.1</td>
<td>0.07</td>
<td>0.35</td>
<td>0.98</td>
<td>3.18</td>
<td></td>
</tr>
</tbody>
</table>
Table 4: Properties of the LSB 20 Galaxy Sample

<table>
<thead>
<tr>
<th>Galaxy</th>
<th>Type</th>
<th>Distance</th>
<th>$L_B$</th>
<th>$R_0$</th>
<th>$R_{last}$</th>
<th>$M_{HI}$</th>
<th>$M_{disk}$</th>
<th>(M/L)$_{stars}$</th>
<th>($v^2/c^2R_{last}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>DDO 0064</td>
<td>LSB</td>
<td>6.8</td>
<td>0.015</td>
<td>1.3</td>
<td>2.1</td>
<td>0.02</td>
<td>0.04</td>
<td>2.87</td>
<td>6.05</td>
</tr>
<tr>
<td>F563-1</td>
<td>LSB</td>
<td>46.8</td>
<td>0.140</td>
<td>2.9</td>
<td>18.2</td>
<td>0.29</td>
<td>1.35</td>
<td>9.65</td>
<td>2.44</td>
</tr>
<tr>
<td>F563-V2</td>
<td>LSB</td>
<td>57.8</td>
<td>0.266</td>
<td>2.0</td>
<td>6.3</td>
<td>0.20</td>
<td>0.60</td>
<td>2.26</td>
<td>6.15</td>
</tr>
<tr>
<td>F568-3</td>
<td>LSB</td>
<td>80.0</td>
<td>0.351</td>
<td>4.2</td>
<td>11.6</td>
<td>0.30</td>
<td>1.20</td>
<td>3.43</td>
<td>3.16</td>
</tr>
<tr>
<td>F583-1</td>
<td>LSB</td>
<td>32.4</td>
<td>0.064</td>
<td>1.6</td>
<td>14.1</td>
<td>0.18</td>
<td>0.15</td>
<td>2.32</td>
<td>1.92</td>
</tr>
<tr>
<td>F583-4</td>
<td>LSB</td>
<td>50.8</td>
<td>0.096</td>
<td>2.8</td>
<td>7.0</td>
<td>0.06</td>
<td>0.31</td>
<td>3.25</td>
<td>2.52</td>
</tr>
<tr>
<td>NGC 0959</td>
<td>LSB</td>
<td>13.5</td>
<td>0.333</td>
<td>1.3</td>
<td>2.9</td>
<td>0.05</td>
<td>0.37</td>
<td>1.11</td>
<td>7.43</td>
</tr>
<tr>
<td>NGC 4395</td>
<td>LSB</td>
<td>4.1</td>
<td>0.374</td>
<td>2.7</td>
<td>0.9</td>
<td>0.13</td>
<td>0.83</td>
<td>2.21</td>
<td>2.29</td>
</tr>
<tr>
<td>NGC 7137</td>
<td>LSB</td>
<td>25.0</td>
<td>0.959</td>
<td>1.7</td>
<td>3.6</td>
<td>0.10</td>
<td>0.27</td>
<td>0.28</td>
<td>3.91</td>
</tr>
<tr>
<td>UGC 0128</td>
<td>LSB</td>
<td>64.6</td>
<td>0.597</td>
<td>6.9</td>
<td>54.8</td>
<td>0.73</td>
<td>2.75</td>
<td>4.60</td>
<td>1.03</td>
</tr>
<tr>
<td>UGC 0191</td>
<td>LSB</td>
<td>15.9</td>
<td>0.129</td>
<td>1.7</td>
<td>2.2</td>
<td>0.26</td>
<td>0.49</td>
<td>3.81</td>
<td>15.48</td>
</tr>
<tr>
<td>UGC 0477</td>
<td>LSB</td>
<td>35.8</td>
<td>0.871</td>
<td>3.5</td>
<td>10.2</td>
<td>1.02</td>
<td>1.00</td>
<td>1.14</td>
<td>4.42</td>
</tr>
<tr>
<td>UGC 1230</td>
<td>LSB</td>
<td>54.1</td>
<td>0.366</td>
<td>4.7</td>
<td>37.1</td>
<td>0.65</td>
<td>0.67</td>
<td>1.82</td>
<td>0.97</td>
</tr>
<tr>
<td>UGC 1281</td>
<td>LSB</td>
<td>5.1</td>
<td>0.017</td>
<td>1.6</td>
<td>1.7</td>
<td>0.03</td>
<td>0.01</td>
<td>0.53</td>
<td>3.02</td>
</tr>
<tr>
<td>UGC 1551</td>
<td>LSB</td>
<td>35.6</td>
<td>0.780</td>
<td>4.2</td>
<td>6.6</td>
<td>0.44</td>
<td>0.16</td>
<td>0.20</td>
<td>3.69</td>
</tr>
<tr>
<td>UGC 4325</td>
<td>LSB</td>
<td>11.9</td>
<td>0.373</td>
<td>1.9</td>
<td>3.4</td>
<td>0.10</td>
<td>0.40</td>
<td>1.08</td>
<td>7.39</td>
</tr>
<tr>
<td>UGC 5005</td>
<td>LSB</td>
<td>51.4</td>
<td>0.200</td>
<td>4.6</td>
<td>27.7</td>
<td>0.28</td>
<td>1.02</td>
<td>5.11</td>
<td>1.30</td>
</tr>
<tr>
<td>UGC 5750</td>
<td>LSB</td>
<td>56.1</td>
<td>0.472</td>
<td>3.3</td>
<td>8.6</td>
<td>0.10</td>
<td>0.10</td>
<td>0.21</td>
<td>1.58</td>
</tr>
<tr>
<td>UGC 5999</td>
<td>LSB</td>
<td>44.9</td>
<td>0.170</td>
<td>4.4</td>
<td>15.0</td>
<td>0.18</td>
<td>3.36</td>
<td>19.81</td>
<td>5.79</td>
</tr>
<tr>
<td>UGC 11820</td>
<td>LSB</td>
<td>17.1</td>
<td>0.169</td>
<td>3.6</td>
<td>3.7</td>
<td>0.40</td>
<td>1.68</td>
<td>9.95</td>
<td>8.44</td>
</tr>
</tbody>
</table>
Table 5: Properties of the LSB 21 Galaxy Sample

<table>
<thead>
<tr>
<th>Galaxy</th>
<th>Type</th>
<th>Distance (Mpc)</th>
<th>$L_B$ ($10^{10}L_\odot$)</th>
<th>$R_0$ (kpc)</th>
<th>$R_{last}$ (kpc)</th>
<th>$M_{HI}$ ($10^{10}M_\odot$)</th>
<th>$M_{disk}$ ($10^{10}M_\odot$)</th>
<th>$(M/L)<em>{stars}$ ($M</em>\odot/L_\odot$)</th>
<th>$(v^2/c^2R)_{last}$ ($10^{-30}$cm$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>ESO 0140040</td>
<td>LSB</td>
<td>217.8</td>
<td>7.169</td>
<td>10.1</td>
<td>30.0</td>
<td>20.70</td>
<td>3.38</td>
<td>8.29</td>
<td>1.49</td>
</tr>
<tr>
<td>ESO 0840411</td>
<td>LSB</td>
<td>82.4</td>
<td>0.287</td>
<td>3.5</td>
<td>9.1</td>
<td>0.06</td>
<td>0.21</td>
<td>0.66</td>
<td>1.19</td>
</tr>
<tr>
<td>ESO 1200211</td>
<td>LSB</td>
<td>15.2</td>
<td>0.028</td>
<td>2.0</td>
<td>3.5</td>
<td>0.01</td>
<td>0.20</td>
<td>0.66</td>
<td>0.66</td>
</tr>
<tr>
<td>ESO 1870510</td>
<td>LSB</td>
<td>16.8</td>
<td>0.054</td>
<td>2.1</td>
<td>2.8</td>
<td>0.09</td>
<td>1.62</td>
<td>2.02</td>
<td>2.02</td>
</tr>
<tr>
<td>ESO 2060140</td>
<td>LSB</td>
<td>59.6</td>
<td>0.735</td>
<td>5.1</td>
<td>11.6</td>
<td>3.51</td>
<td>4.78</td>
<td>4.34</td>
<td>4.34</td>
</tr>
<tr>
<td>ESO 3020120</td>
<td>LSB</td>
<td>70.9</td>
<td>0.717</td>
<td>3.4</td>
<td>11.2</td>
<td>0.77</td>
<td>1.07</td>
<td>2.37</td>
<td>2.37</td>
</tr>
<tr>
<td>ESO 3050090</td>
<td>LSB</td>
<td>13.2</td>
<td>0.186</td>
<td>1.3</td>
<td>5.6</td>
<td>0.06</td>
<td>0.32</td>
<td>1.87</td>
<td>1.87</td>
</tr>
<tr>
<td>ESO 4250180</td>
<td>LSB</td>
<td>88.3</td>
<td>2.600</td>
<td>7.3</td>
<td>14.6</td>
<td>4.79</td>
<td>1.84</td>
<td>5.17</td>
<td>5.17</td>
</tr>
<tr>
<td>ESO 4880490</td>
<td>LSB</td>
<td>28.7</td>
<td>0.139</td>
<td>1.6</td>
<td>7.8</td>
<td>0.43</td>
<td>3.07</td>
<td>4.34</td>
<td>4.34</td>
</tr>
<tr>
<td>F571-8</td>
<td>LSB</td>
<td>50.3</td>
<td>0.191</td>
<td>5.4</td>
<td>14.6</td>
<td>0.16</td>
<td>4.48</td>
<td>23.49</td>
<td>5.10</td>
</tr>
<tr>
<td>F579-V1</td>
<td>LSB</td>
<td>86.9</td>
<td>0.557</td>
<td>5.2</td>
<td>14.7</td>
<td>0.21</td>
<td>3.33</td>
<td>5.98</td>
<td>3.18</td>
</tr>
<tr>
<td>F730-V1</td>
<td>LSB</td>
<td>148.3</td>
<td>0.756</td>
<td>5.8</td>
<td>12.2</td>
<td>5.95</td>
<td>7.87</td>
<td>6.22</td>
<td>6.22</td>
</tr>
<tr>
<td>UGC 04115</td>
<td>LSB</td>
<td>5.5</td>
<td>0.004</td>
<td>0.3</td>
<td>1.7</td>
<td>0.01</td>
<td>0.97</td>
<td>3.42</td>
<td>3.42</td>
</tr>
<tr>
<td>UGC 06614</td>
<td>LSB</td>
<td>86.2</td>
<td>2.109</td>
<td>8.2</td>
<td>62.7</td>
<td>2.07</td>
<td>9.70</td>
<td>4.60</td>
<td>2.39</td>
</tr>
<tr>
<td>UGC 11454</td>
<td>LSB</td>
<td>93.9</td>
<td>0.456</td>
<td>3.4</td>
<td>12.3</td>
<td>3.15</td>
<td>6.90</td>
<td>6.79</td>
<td>6.79</td>
</tr>
<tr>
<td>UGC 11557</td>
<td>LSB</td>
<td>23.7</td>
<td>1.806</td>
<td>3.0</td>
<td>6.7</td>
<td>0.25</td>
<td>0.37</td>
<td>0.20</td>
<td>3.49</td>
</tr>
<tr>
<td>UGC 11583</td>
<td>LSB</td>
<td>7.1</td>
<td>0.012</td>
<td>0.7</td>
<td>2.1</td>
<td>0.01</td>
<td>0.96</td>
<td>2.15</td>
<td>2.15</td>
</tr>
<tr>
<td>UGC 11616</td>
<td>LSB</td>
<td>74.9</td>
<td>2.159</td>
<td>3.1</td>
<td>9.8</td>
<td>2.43</td>
<td>1.13</td>
<td>7.49</td>
<td>7.49</td>
</tr>
<tr>
<td>UGC 11648</td>
<td>LSB</td>
<td>49.0</td>
<td>4.073</td>
<td>4.0</td>
<td>13.0</td>
<td>2.57</td>
<td>0.63</td>
<td>5.79</td>
<td>5.79</td>
</tr>
<tr>
<td>UGC 11748</td>
<td>LSB</td>
<td>75.3</td>
<td>23.930</td>
<td>2.6</td>
<td>21.6</td>
<td>9.67</td>
<td>0.40</td>
<td>1.01</td>
<td>1.01</td>
</tr>
<tr>
<td>UGC 11819</td>
<td>LSB</td>
<td>61.5</td>
<td>2.155</td>
<td>4.7</td>
<td>11.9</td>
<td>4.83</td>
<td>2.24</td>
<td>7.03</td>
<td>7.03</td>
</tr>
</tbody>
</table>
Table 6: Properties of the Miscellaneous 21 Galaxy Sample

<table>
<thead>
<tr>
<th>Galaxy</th>
<th>Type</th>
<th>Distance (Mpc)</th>
<th>$L_B$ ($10^{10}L_\odot$)</th>
<th>$R_0$ (kpc)</th>
<th>$R_{last}$ (kpc)</th>
<th>$M_{HI}$ ($10^{10}M_\odot$)</th>
<th>$M_{disk}$ ($10^{10}M_\odot$)</th>
<th>$(M/L)<em>\text{stars}$ ($M</em>\odot/L_\odot$)</th>
<th>$(v^2/c^2R)_{last}$ ($10^{-30}$cm$^{-1}$)</th>
<th>$v$</th>
</tr>
</thead>
<tbody>
<tr>
<td>DDO 0168</td>
<td>LSB</td>
<td>4.5</td>
<td>0.032</td>
<td>1.2</td>
<td>4.4</td>
<td>0.03</td>
<td>0.06</td>
<td>2.03</td>
<td>2.22</td>
<td></td>
</tr>
<tr>
<td>DDO 0170</td>
<td>LSB</td>
<td>16.6</td>
<td>0.023</td>
<td>1.9</td>
<td>13.3</td>
<td>0.09</td>
<td>0.05</td>
<td>1.97</td>
<td>1.18</td>
<td></td>
</tr>
<tr>
<td>M 0033</td>
<td>HSB</td>
<td>0.9</td>
<td>0.850</td>
<td>2.5</td>
<td>8.9</td>
<td>0.11</td>
<td>1.13</td>
<td>1.33</td>
<td>4.62</td>
<td></td>
</tr>
<tr>
<td>NGC 0055</td>
<td>LSB</td>
<td>1.9</td>
<td>0.588</td>
<td>1.9</td>
<td>12.2</td>
<td>0.13</td>
<td>0.30</td>
<td>0.50</td>
<td>2.22</td>
<td></td>
</tr>
<tr>
<td>NGC 0247</td>
<td>LSB</td>
<td>3.6</td>
<td>0.512</td>
<td>4.2</td>
<td>14.3</td>
<td>0.16</td>
<td>1.25</td>
<td>2.43</td>
<td>2.94</td>
<td></td>
</tr>
<tr>
<td>NGC 0300</td>
<td>LSB</td>
<td>2.0</td>
<td>0.271</td>
<td>2.1</td>
<td>11.7</td>
<td>0.08</td>
<td>0.65</td>
<td>2.41</td>
<td>2.69</td>
<td></td>
</tr>
<tr>
<td>NGC 0801</td>
<td>HSB</td>
<td>63.0</td>
<td>4.746</td>
<td>9.5</td>
<td>46.7</td>
<td>1.39</td>
<td>6.93</td>
<td>2.37</td>
<td>3.59</td>
<td></td>
</tr>
<tr>
<td>NGC 1003</td>
<td>LSB</td>
<td>11.8</td>
<td>1.480</td>
<td>1.9</td>
<td>31.2</td>
<td>0.63</td>
<td>0.66</td>
<td>0.45</td>
<td>1.53</td>
<td></td>
</tr>
<tr>
<td>NGC 1560</td>
<td>LSB</td>
<td>3.7</td>
<td>0.053</td>
<td>1.6</td>
<td>10.3</td>
<td>0.12</td>
<td>0.17</td>
<td>3.16</td>
<td>2.16</td>
<td></td>
</tr>
<tr>
<td>NGC 2683</td>
<td>HSB</td>
<td>10.2</td>
<td>1.882</td>
<td>2.4</td>
<td>36.0</td>
<td>0.15</td>
<td>6.03</td>
<td>3.20</td>
<td>2.28</td>
<td></td>
</tr>
<tr>
<td>NGC 2998</td>
<td>HSB</td>
<td>59.3</td>
<td>5.186</td>
<td>4.8</td>
<td>41.1</td>
<td>1.78</td>
<td>7.16</td>
<td>1.75</td>
<td>3.43</td>
<td></td>
</tr>
<tr>
<td>NGC 3109</td>
<td>LSB</td>
<td>1.5</td>
<td>0.064</td>
<td>1.3</td>
<td>7.1</td>
<td>0.06</td>
<td>0.02</td>
<td>0.35</td>
<td>2.29</td>
<td></td>
</tr>
<tr>
<td>NGC 5033</td>
<td>LSB</td>
<td>15.3</td>
<td>3.058</td>
<td>7.5</td>
<td>45.6</td>
<td>1.07</td>
<td>0.27</td>
<td>3.28</td>
<td>3.16</td>
<td></td>
</tr>
<tr>
<td>NGC 5371</td>
<td>HSB</td>
<td>35.3</td>
<td>7.593</td>
<td>4.4</td>
<td>41.0</td>
<td>0.89</td>
<td>8.52</td>
<td>1.44</td>
<td>3.98</td>
<td></td>
</tr>
<tr>
<td>NGC 5533</td>
<td>HSB</td>
<td>42.0</td>
<td>3.173</td>
<td>7.4</td>
<td>56.0</td>
<td>1.39</td>
<td>2.00</td>
<td>4.14</td>
<td>3.31</td>
<td></td>
</tr>
<tr>
<td>NGC 5585</td>
<td>LSB</td>
<td>9.0</td>
<td>0.333</td>
<td>2.0</td>
<td>14.0</td>
<td>0.28</td>
<td>0.36</td>
<td>1.09</td>
<td>2.06</td>
<td></td>
</tr>
<tr>
<td>NGC 5907</td>
<td>HSB</td>
<td>16.5</td>
<td>5.400</td>
<td>5.5</td>
<td>48.0</td>
<td>1.90</td>
<td>2.49</td>
<td>1.89</td>
<td>3.44</td>
<td></td>
</tr>
<tr>
<td>NGC 6503</td>
<td>HSB</td>
<td>5.5</td>
<td>0.417</td>
<td>1.6</td>
<td>20.7</td>
<td>0.14</td>
<td>1.53</td>
<td>3.66</td>
<td>2.30</td>
<td></td>
</tr>
<tr>
<td>NGC 6674</td>
<td>LSB</td>
<td>42.0</td>
<td>4.935</td>
<td>7.1</td>
<td>59.1</td>
<td>2.18</td>
<td>2.00</td>
<td>2.52</td>
<td>3.57</td>
<td></td>
</tr>
<tr>
<td>UGC 2259</td>
<td>LSB</td>
<td>10.0</td>
<td>0.110</td>
<td>1.4</td>
<td>7.8</td>
<td>0.04</td>
<td>0.47</td>
<td>4.23</td>
<td>3.76</td>
<td></td>
</tr>
<tr>
<td>UGC 2885</td>
<td>HSB</td>
<td>80.4</td>
<td>23.955</td>
<td>13.3</td>
<td>74.1</td>
<td>3.98</td>
<td>8.47</td>
<td>0.72</td>
<td>4.31</td>
<td></td>
</tr>
</tbody>
</table>
FIG. 1: Fitting to the rotational velocities (in km sec$^{-1}$) of the THINGS 18 galaxy sample.
FIG. 2: Fitting to the rotational velocities of the Ursa Major 30 galaxy sample – Part 1
FIG. 2: Fitting to the rotational velocities of the Ursa Major 30 galaxy sample – Part 2
FIG. 3: Fitting to the rotational velocities of the LSB 20 galaxy sample
FIG. 4: Fitting to the rotational velocities of the LSB 21 galaxy sample
FIG. 5: Fitting to the rotational velocities of the Miscellaneous 21 galaxy sample
FIG. 6: Extended distance predictions for DDO 154 and UGC 128.