AdS/CFT, Minimal Unitary Representations and Spectrum Generating Symmetries

Murat Günaydin

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- AdS/CFT and minimal unitary representations of noncompact groups and supergroups
- Open problems
Bosonic part of the 5D Maxwell-Einstein supergravity Lagrangian (MESGT)

\[ e^{-1} L = -\frac{1}{2} R - \frac{1}{4} \partial_\mu \partial_\nu F^I_{\mu\nu} F^J_{\mu\nu} - \frac{1}{2} g_{xy} (\partial_\mu \varphi^x) (\partial_\nu \varphi^y) + \]
\[ + \frac{e^{-1}}{6\sqrt{6}} C_{IJK} \varepsilon^{\mu\nu\rho\sigma\lambda} F^I_{\mu\nu} F^J_{\rho\sigma} A^K_\lambda \]

describing the coupling of \((n_V - 1)\) vector multiplets \((A_\mu, \lambda^i, \varphi)\) to \(N = 2\) supergravity \((g_{\mu\nu}, \psi^i_\mu, A_\mu)\).

\(I = 1, \ldots, n_V\), \(x = 1, \ldots, (n_V - 1)\)
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\(N = 2\) MESGT is uniquely determined by the constant symmetric tensor \(C_{IJK}\).

5D MESGTs with symmetric scalar manifolds \(G/H\) such that \(G\) is a symmetry of the Lagrangian \(\implies C_{IJK}\) is given by the norm (determinant) \(N_{3}\) of a Euclidean Jordan algebra \(J\) of degree 3.

\[ N_{3}(J) = C_{IJK} h^{I} h^{J} h^{K} \]
There exist only four unified $N = 2$ Maxwell-Einstein Supergravity theories in five dimensions with symmetric scalar manifolds. They are defined by the four simple Jordan algebras $J_3^A$ of $3 \times 3$ Hermitian matrices over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$. These *magical* supergravity theories describe the coupling of 5, 8, 14 and 26 vector multiplets to supergravity.
There exist only four unified $N = 2$ Maxwell-Einstein Supergravity theories in five dimensions with symmetric scalar manifolds. They are defined by the four simple Jordan algebras $J^A_3$ of $3 \times 3$ Hermitian matrices over $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$. These *magical* supergravity theories describe the coupling of 5, 8, 14 and 26 vector multiplets to supergravity.

Their scalar manifolds are the irreducible symmetric spaces

\begin{align*}
J^R_3 : \quad & M_5 = SL(3, \mathbb{R})/SO(3) \\
J^C_3 : \quad & M_5 = SL(3, \mathbb{C})/SU(3) \\
J^H_3 : \quad & M_5 = SU^*(6)/USp(6) \\
J^O_3 : \quad & M_5 = E_6(-26)/F_4
\end{align*}

The cubic norm form, $N_3$, of the simple Jordan algebras is given by the determinant of the corresponding Hermitian $(3 \times 3)$-matrices.
Exceptional $N = 2$ versus Maximal $N = 8$ Supergravity:

- The exceptional $N = 2$ supergravity is defined by the exceptional Jordan algebra $J_3^{\mathbb{O}}$ of $3 \times 3$ Hermitian matrices over real octonions. Its global invariance group in 5D is $E_6(-26)$ with maximal compact subgroup $F_4$. 

- The $C$-tensor $C_{IJK}$ of $N = 8$ supergravity in five dimensions can be identified with the symmetric tensor given by the cubic norm of the split exceptional Jordan algebra $J_3^{\mathbb{O}_s}$ defined over split octonions $\mathbb{O}_s$. Its global invariance group in 5D is $E_6(6)$ with maximal compact subgroup $\text{USp}(8)$.

- In $D = 4$ and $D = 3$ the exceptional supergravity has $E_7(-25)$ and $E_8(-24)$ as its U-duality group while the maximal $N = 8$ supergravity has $E_7(7)$ and $E_8(8)$, respectively.
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Symmetry Groups of Simple Jordan algebras of Arbitrary Rank

<table>
<thead>
<tr>
<th>$J$</th>
<th>$\text{Rot}(J)$</th>
<th>$\text{Lor}(J)$</th>
<th>$\text{Conf}(J)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_n^\mathbb{R}$</td>
<td>$SO(n)$</td>
<td>$SL(n, \mathbb{R})$</td>
<td>$Sp(2n, \mathbb{R})$</td>
</tr>
<tr>
<td>$J_n^\mathbb{C}$</td>
<td>$SU(n)$</td>
<td>$SL(n, \mathbb{C})$</td>
<td>$SU(n, n)$</td>
</tr>
<tr>
<td>$J_n^\mathbb{H}$</td>
<td>$USp(2n)$</td>
<td>$SU^*(2n)$</td>
<td>$SO^*(4n)$</td>
</tr>
<tr>
<td>$J^\text{O}_3$</td>
<td>$F_4$</td>
<td>$E_6(-26)$</td>
<td>$E_7(-25)$</td>
</tr>
<tr>
<td>$\Gamma_{(1,d)}$</td>
<td>$SO(d)$</td>
<td>$SO(d, 1)$</td>
<td>$SO(d, 2)$</td>
</tr>
</tbody>
</table>

**Table:** Above we give the complete list of simple Euclidean Jordan algebras and their rotation (automorphism), Lorentz (reduced structure) and Conformal (linear fractional) groups.
The black hole potential that determines the attractor flow takes on the following form for $N = 2$ MESGTs: (Ferrara, Gibbons, Kallosh, Strominger)

$$V(\phi, q) = q_I \circ a^{IJ} q_J$$

where $\circ a^{IJ}$ is the "metric" of the kinetic energy term of the vector fields. The $(n+1)$ dimensional charge vector in an extremal BH background is given by

$$q_I = \int_{S^3} H_I = \int_{S^3} \circ a_{IJ} * F^J \quad (I = 0, 1, \ldots n)$$

The entropy $S$ of an extremal black hole solution of $N = 2$ MESGT with charges $q_I$ is determined by the value of the black hole potential $V$ at the attractor points

$$S_{BPS} = (V_{critical})^{3/4} = \left(C^{IJK} q_I q_J q_K\right)^{3/4}$$
Table: Orbits of spherically symmetric stationary BPS black hole solutions in 5D MESGTs defined by Euclidean Jordan algebras $J$ of degree three. U-duality and stability groups are given by the Lorentz (reduced structure) and rotation (automorphism) groups of $J$.

<table>
<thead>
<tr>
<th>$J$</th>
<th>$\mathcal{O}_{BPS} = \text{Str}_0(J)/\text{Aut}(J)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_3^R$</td>
<td>$SL(3, \mathbb{R})/SO(3)$</td>
</tr>
<tr>
<td>$J_3^C$</td>
<td>$SL(3, \mathbb{C})/SU(3)$</td>
</tr>
<tr>
<td>$J_3^H$</td>
<td>$SU^*(6)/USp(6)$</td>
</tr>
<tr>
<td>$J_3^O$</td>
<td>$E_6(-26)/F_4$</td>
</tr>
<tr>
<td>$\mathbb{R} \oplus \Gamma_{(1,n-1)}$</td>
<td>$SO(n - 1, 1) \times SO(1, 1)/SO(n - 1)$</td>
</tr>
</tbody>
</table>

Table: Orbits of non-BPS extremal BHs with non-zero entropy.
The orbits of BPS black hole solutions of $N=8$ supergravity theory in five dimensions.

The 1/8 BPS black holes with non-vanishing entropy has the orbit

$$\mathcal{O}_{1/8-BPS} = \frac{E_{6(6)}}{F_{4(4)}}$$

Maximal supergravity theory admits 1/4 and 1/2 BPS black hole solutions with vanishing entropy. Their orbits under U-duality are

$$\mathcal{O}_{1/4-BPS} = \frac{E_{6(6)}}{O(5,4) \mathbb{S} T_{16}}$$

$$\mathcal{O}_{1/2-BPS} = \frac{E_{6(6)}}{O(5,5) \mathbb{S} T_{16}}$$

Vanishing entropy means vanishing cubic norm. Thus the black hole solutions corresponding to vanishing entropy has additional symmetries beyond the five dimensional U-duality group.

$\Rightarrow$ They are invariant under the generalized special conformal transformations of the underlying Jordan algebras.
Hence the proposal that the conformal groups $Conf[J]$ of underlying Jordan algebras $J$ of supergravity theories must act as spectrum generating symmetry groups. $Conf[J]$ leaves invariant light-like separations with respect to a cubic distance function $N_3(J_1 - J_2)$ and has a natural 3-grading with respect to their Lorentz subgroups

$$Conf[J] = K_J \oplus Lor(J) \times D \oplus T_J$$

$Lor(J)$ is the 5D U-duality group that leaves the cubic norm invariant.
Hence the proposal that the conformal groups \( \text{Conf}[J] \) of underlying Jordan algebras \( J \) of supergravity theories must act as spectrum generating symmetry groups. \( \text{Conf}[J] \) leaves invariant light-like separations with respect to a cubic distance function \( \mathcal{N}_3(J_1 - J_2) \) and has a natural 3-grading with respect to their Lorentz subgroups

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\( \text{Conf}[J] \Leftrightarrow \) U-duality group of corresponding 4D supergravity.
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$\text{Conf}[J] \Leftrightarrow$ U-duality group of corresponding 4$D$ supergravity.

4$D$ U-duality groups must act as spectrum generating symmetry groups of corresponding five dimensional supergravity theories.
Classification of the orbits of spherically symmetric stationary extremal black holes of $4D$, $N = 8$ sugra and $4D$, $N = 2$ MESGTs with symmetric target spaces. (Ferrara and MG, 1997).
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Question: Can the 3D U-duality groups act as spectrum generating conformal symmetries of corresponding 4D supergravity theories? (MG, Koepsell, Nicolai)
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No conformal realization for any real forms of $E_8$, $G_2$ and $F_4$ ⇔ No 3-grading with respect to a subgroup of maximal rank.
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No conformal realization for any real forms of $E_8, G_2$ and $F_4$ \iff No 3-grading with respect to a subgroup of maximal rank.

However, all simple Lie algebras admit a 5-grading with respect to a subalgebra of maximal rank

$$g = g^{-2} \oplus g^{-1} \oplus g^0 \oplus g^1 \oplus g^2$$

such that the grade $\pm 2$ subspaces are one-dimensional.
Under dimensional reduction of $5D$ MESGT to $4D$

$$\mathcal{M}_5 = \frac{\text{Lor}(J)}{\text{Rot}(J)} \Rightarrow \mathcal{M}_4 = \frac{\text{Conf}(J)}{\text{Lor}(J) \times U(1)}$$

where $\text{Lor}(J)$ is the compact real form of the Lorentz group of $J$. The bosonic sector of dimensionally reduced Lagrangian is

$$e^{-1}L^{(4)} = -\frac{1}{2}R - g_{i\bar{j}}(\partial_\mu z^i)(\partial_\mu \bar{z}^\bar{j})$$

$$+ \frac{1}{4}\text{Im}(\mathcal{N}_{AB}) F^A_{\mu\nu} F^{\mu\nu B} - \frac{1}{8}\text{Re}(\mathcal{N}_{AB}) \epsilon^{\mu\nu\rho\sigma} F^A_{\mu\nu} F^B_{\rho\sigma}$$

In $5D$: Vector fields $A^\mu_i \Leftrightarrow$ Elements of Jordan algebra $J$

In $4D$: $F^A_{\mu\nu} \oplus \tilde{F}^A_{\mu\nu} \Leftrightarrow$ Elements of a Freudenthal triple system (FTS) $\mathcal{F}(J)$:

$$X = \begin{bmatrix} R & J & F^0_{\mu\nu} & F^I_{\mu\nu} \\ J & R & \tilde{F}^I_{\mu\nu} & \tilde{F}^0_{\mu\nu} \end{bmatrix}$$
U-duality group $G_4$ of a 4D Maxwell-Einstein supergravity $\Leftrightarrow Aut(\mathcal{F}(J)) \equiv Conf[J]$

The entropy of an extremal black hole with charges $(p^0, p^I, q_0, q_I)$ is given by the quartic invariant $Q_4(q, p)$ of $\mathcal{F}(J)$.

black hole attractor equations $\Rightarrow$ criticality conditions for black hole scalar potential

$$V_{BH} \equiv |Z|^2 + G^{I\bar{J}}(D_I Z)(\bar{D}_{\bar{J}} \bar{Z})$$

$Z \equiv$ central charge function.

$$\partial_I V_{BH} = 0$$

implies

$$2\bar{Z} D_I Z + iC_{IJK} G^{J\bar{J}} G^{K\bar{K}} \bar{D}_{\bar{J}} \bar{Z} \bar{D}_{\bar{K}} \bar{Z} = 0$$
<table>
<thead>
<tr>
<th>J</th>
<th>$\frac{1}{2}$-BPS orbits</th>
<th>non-BPS, $Z \neq 0$ orbits</th>
<th>non-BPS, $Z = 0$ orbits</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$O_{\frac{1}{2} - BPS}$</td>
<td>$O_{\text{non-BPS}, Z \neq 0}$</td>
<td>$O_{\text{non-BPS}, Z = 0}$</td>
</tr>
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<td>$\mathbb{R} \oplus \Gamma(1,n-1)$</td>
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<td></td>
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<td>$E_7(-25)/E_6$</td>
<td>$E_7(-25)/E_6(-26)$</td>
<td>$E_7(-25)/E_6(-14)$</td>
</tr>
<tr>
<td>$J^\mathbb{H}_3$</td>
<td>$SO^*(12)/SU(6)$</td>
<td>$SO^<em>(12)/SU^</em>(6)$</td>
<td>$SO^*(12)/SU(4,2)$</td>
</tr>
<tr>
<td>$J^\mathbb{C}_3$</td>
<td>$SU(3,3)/SU(1,2)$</td>
<td>$SU(3,3)/SU(1,2)$</td>
<td>$SU(3,3)/SU(1,2)$</td>
</tr>
<tr>
<td>$J^\mathbb{R}_3$</td>
<td>$Sp(6,R)/SU(3)$</td>
<td>$Sp(6,R)/SL(3,R)$</td>
<td>$Sp(6,R)/SU(2,1)$</td>
</tr>
</tbody>
</table>

**Table:** Non-degenerate orbits of $N = 2$, $D = 4$ MESGTs with symmetric scalar manifolds. Except for the first row all such theories originate from five dimensions and are defined by Jordan algebras that are indicated in the first column.

Bellucci, Ferrara, MG, Marrani
The orbits of BH solutions of 4D \( N = 8 \) supergravity under \( E_7(7) \):

\[
I_4 > 0 : \mathcal{O}_{\frac{1}{8}-BPS} = \frac{E_7(7)}{E_6(2)} \iff \frac{1}{8}\text{-BPS;}
\]

\[
I_4 < 0 : \mathcal{O}_{\text{non-BPS}} = \frac{E_7(7)}{E_6(6)} \iff \text{non-BPS}.
\]

Generic light-like orbit with 1 vanishing eigenvalue:

\[
\frac{E_7(7)}{F_4(4) \oplus T_26}
\]

Critical light-like orbit with 2 vanishing eigenvalues:

\[
\frac{E_7(7)}{O(6,5) \oplus (T_{32} \oplus T_1)}
\]

Doubly critical light-like orbit with 3 vanishing eigenvalues:

\[
\frac{E_7(7)}{E_6(6) \oplus T_{27}}
\]
QUASICONFORMAL REALIZATION OF $E_{8(8)}$ OVER A 57 DIMENSIONAL SPACE

$E_{8(8)} = 1_{-2} \oplus 56_{-1} \oplus E_{7(7)} + SO(1, 1) \oplus 56_{+1} \oplus 1_{+2}$

$g = \tilde{K} \oplus \tilde{U}_A \oplus [S_{(AB)} + \Delta] \oplus U_A \oplus K$

over a space $\mathcal{T}$ coordinatized by the elements $X$ of FTS $\mathcal{F}$ plus an extra singlet variable $x$: $56_{+1} \oplus 1_{+2} \Leftrightarrow (X, x) \in \mathcal{T}$:

$K(X) = 0 \quad U_A(X) = A \quad S_{AB}(X) = (A, B, X)$

$K(x) = 2 \quad U_A(x) = \langle A, X \rangle \quad S_{AB}(x) = 2 \langle A, B \rangle x$

$\tilde{U}_A(X) = \frac{1}{2} (X, A, X) - Ax$

$\tilde{U}_A(x) = -\frac{1}{6} \langle (X, X, X), A \rangle + \langle X, A \rangle x$

$\tilde{K}(X) = -\frac{1}{6} (X, X, X) + Xx$

$\tilde{K}(x) = \frac{1}{6} \langle (X, X, X), X \rangle + 2 x^2$

FTP $\Leftrightarrow (X, Y, Z)$ & Symplectic form $\Leftrightarrow \langle X, Y \rangle = -\langle Y, X \rangle$

M. Günyaydin, Miami 2010
Geometric meaning of the quasiconformal action of the Lie algebra $\mathfrak{g}$ on the space $\mathcal{T}$?

Define a quartic norm of $X = (X, x) \in \mathcal{T}$ as $N^4(X) := Q^4(X) - x^2$. $Q^4(X)$ is the quartic norm of $X \in F$.

Define a quartic "distance" function between any two points $X = (X, x)$ and $Y = (Y, y)$ in $\mathcal{T}$ as $d(X, Y) := N^4(\delta(X, Y)) \delta(X, Y)$ is the "symplectic" difference of $X$ and $Y$:

$\delta(X, Y) := (X - Y, x - y + \langle X, Y \rangle) = -\delta(Y, X)$.

Light-like separations $d(X, Y) = 0$ are left invariant under quasiconformal group action. Quasiconformal groups are the invariance groups of "light-cones" defined by a quartic distance function. $E_8(8)$ is the invariance group of a quartic light-cone in 57 dimensions! M. G"unaydin, Miami 2010 17
Geometric meaning of the quasiconformal action of the Lie algebra $\mathfrak{g}$ on the space $\mathcal{T}$?

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$$\mathcal{N}_4(X) := Q_4(X) - x^2$$

$Q_4(X)$ is the quartic norm of $X \in \mathcal{F}$.

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Quasiconformal groups are the invariance groups of "light-cones" defined by a quartic distance function.

$E_8(8)$ is the invariance group of a quartic light-cone in 57 dimensions!
Geometric meaning of the quasiconformal action of the Lie algebra $g$ on the space $\mathcal{T}$?

Define a quartic norm of $\mathcal{X} = (X, x) \in \mathcal{T}$ as

$$N_4(\mathcal{X}) := Q_4(X) - x^2$$

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Geometric meaning of the quasiconformal action of the Lie algebra \( g \) on the space \( \mathcal{T} \)?

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\]
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*Quasiconformal groups are the invariance groups of "light-cones" defined by a quartic distance function.*

\( E_{8(8)} \) is the invariance group of a quartic light-cone in 57 dimensions!
Scalar manifolds of $N = 2$ MESGTs defined by Jordan algebras of degree three (8 real supersymmetries)

\[ \mathcal{M}_5 = \frac{\text{Lor}(J)}{\text{Rot}(J)} \]

\[ \mathcal{M}_4 = \frac{\text{Conf}(J)}{\text{Lor}(J) \times U(1)} \]

\[ \mathcal{M}_3 = \frac{\text{QConf}(J)}{\text{Conf}(J) \times SU(2)} \]

$\text{QConf}(J)$ is the quasiconformal group associated with $J$. Tilde $\sim$ denotes the compact form.
Scalar manifolds of $N = 2$ MESGTs defined by Jordan algebras of degree three (8 real supersymmetries)

\[ M_5 = \frac{\text{Lor}(J)}{\text{Rot}(J)} \]

\[ M_4 = \frac{\text{Conf}(J)}{\text{Lor}(J) \times U(1)} \]

\[ M_3 = \frac{\text{QConf}(J)}{\text{Conf}(J) \times SU(2)} \]

$\text{QConf}(J)$ is the quasiconformal group associated with $J$. Tilde $\sim$ denotes the compact form.

Quasiconformal extensions of 4D U-duality groups $\equiv$ 3D U-duality groups.

Proposal: 3D U-duality groups must act as spectrum generating symmetry groups of the extremal black hole solutions of 4D supergravity theories.

GKN 2000
<table>
<thead>
<tr>
<th>$J$</th>
<th>$\mathcal{M}_5 = \text{Lor} (J) / \text{Rot} (J)$</th>
<th>$\mathcal{M}_4 = \text{Conf} (J) / \widetilde{\text{Lor}} (J) \times U(1)$</th>
<th>$\mathcal{M}_3 = \widetilde{\text{QConf}} (\mathcal{F} (J)) / \text{Conf} (J) \times SU(2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_3^R$</td>
<td>$\text{SL}(3, \mathbb{R})/\text{SO}(3)$</td>
<td>$\text{Sp}(6, \mathbb{R})/U(3)$</td>
<td>$\text{F}_4(4)/USp(6) \times SU(2)$</td>
</tr>
<tr>
<td>$J_3^C$</td>
<td>$\text{SL}(3, \mathbb{C})/\text{SU}(3)$</td>
<td>$\text{SU}(3,3)/SU(3) \times SU(3)$</td>
<td>$\text{E}_6(2)/SU(6) \times SU(2)$</td>
</tr>
<tr>
<td>$J_3^H$</td>
<td>$\text{SU}^*(6)/\text{USp}(6)$</td>
<td>$\text{SO}^*(12)/U(6)$</td>
<td>$\text{E}_7(-5)/SO(12) \times SU(2)$</td>
</tr>
<tr>
<td>$J_3^D$</td>
<td>$\text{E}_6(-26)/\text{F}_4$</td>
<td>$\text{E}_7(-25)/\text{E}_6 \times U(1)$</td>
<td>$\text{E}_8(-24)/\text{E}_7 \times SU(2)$</td>
</tr>
<tr>
<td>$\mathbb{R} \oplus \Gamma_{(1,n-1)}$</td>
<td>$\text{SO}(n-1,1) \times \text{SO}(1,1) / \text{SO}(n-1)$</td>
<td>$\text{SO}(n,2) \times \text{SU}(1,1) / \text{SO}(n) \times \text{SO}(2) \times U(1)$</td>
<td>$\text{SO}(n+2,4) / \text{SO}(n+2) \times \text{SO}(4)$</td>
</tr>
</tbody>
</table>

**Table:** Scalar manifolds $\mathcal{M}_d$ of $N = 2$ MESGT's defined by Jordan algebras $J$ of degree 3 in $d = 3, 4, 5$ dimensions. $\text{Lor} (J)$ and $\text{Conf} (J)$ denote the compact real forms of the Lorentz group $\text{Lor} (J)$ and conformal group $\text{Conf} (J)$ of a Jordan algebra $J$. $\widetilde{\text{QConf}} (\mathcal{F} (J))$ denotes the quasiconformal group associated with $J$. 

M. Günaydin, Miami 2010
A concrete implementation of the proposal that 3D U-duality groups must act as spectrum generating quasiconformal groups of spherically symmetric stationary BPS black holes of 4D supergravity theories:

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Equations of motion for a spherically symmetric stationary black hole of four dimensional supergravity theories are equivalent to equations for geodesic motion of a fiducial particle on the moduli space $\mathcal{M}_3^*$ of 3D supergravity obtained by reduction on a time-like circle.

$3D$ scalar manifold from compactification on a space-like circle

$$\mathcal{M}_3 = \frac{G_3}{K_3}$$

where $K_3$ is the maximal compact subgroup of $G_3$. On a time-like circle one gets the scalar manifold

$$\mathcal{M}_3^* = \frac{G_3}{H_3}$$

where $H_3$ is a noncompact real form of $K_3$. 


Breitenlohner, Gibbons, Maison 1987

M. Günaydin, Miami 2010
<table>
<thead>
<tr>
<th>$n_Q$</th>
<th>$n_V$</th>
<th>$M_4$</th>
<th>$\mathcal{M}_3^*$</th>
<th>$J$</th>
</tr>
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<tbody>
<tr>
<td>8</td>
<td>1</td>
<td>$\emptyset$</td>
<td>$U(2,1)$</td>
<td>$\mathbb{R}$</td>
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<td>$SL(2,\mathbb{R})/U(1)$</td>
<td>$G_2,2$</td>
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<tr>
<td>8</td>
<td>7</td>
<td>$Sp(6,\mathbb{R})/SU(3)\times U(1)$</td>
<td>$F_4(4)$</td>
<td>$J_3^R$</td>
</tr>
<tr>
<td>8</td>
<td>10</td>
<td>$SU(3,3)/SU(3)\times SU(3)\times U(1)$</td>
<td>$E_6(2)$</td>
<td>$J_3^C$</td>
</tr>
<tr>
<td>8</td>
<td>16</td>
<td>$SO^*(12)/SU(6)\times U(1)$</td>
<td>$E_7(-25)$</td>
<td>$J_3^H$</td>
</tr>
<tr>
<td>8</td>
<td>28</td>
<td>$E_{7(-25)}/E_6\times U(1)$</td>
<td>$E_8(-24)$</td>
<td>$J_3^O$</td>
</tr>
<tr>
<td>8</td>
<td>$n+2$</td>
<td>$SL(2,\mathbb{R})/U(1) \times SO(n,2)/SO(n)\times SO(2)$</td>
<td>$SO(n+2,4)/SO(n,2)\times SO(2,2)$</td>
<td>$\mathbb{R} \oplus \Gamma_{(1,n-1)}$</td>
</tr>
<tr>
<td>16</td>
<td>$n+2$</td>
<td>$SL(2,\mathbb{R})/U(1) \times SO(n-4,6)/SO(n-4)\times SO(6)$</td>
<td>$SO(n-2,8)/SO(n-4,2)\times SO(2,6)$</td>
<td>$\mathbb{R} \oplus \Gamma_{(5,n-5)}$</td>
</tr>
<tr>
<td>24</td>
<td>16</td>
<td>$SO^*(12)/SU(6)\times U(1)$</td>
<td>$E_7(-5)$</td>
<td>$J_3^H$</td>
</tr>
<tr>
<td>32</td>
<td>28</td>
<td>$E_7(7)/SU(8)$</td>
<td>$E_8(8)$</td>
<td>$J_3^{Os}$</td>
</tr>
</tbody>
</table>

**Table:** Above we give the number of supercharges $n_Q$, 4D vector fields $n_V$, scalar manifolds of supergravity theories before and after reduction along a timelike Killing vector from $D = 4$ to $D = 3$, and associated Jordan algebras $J$. Isometry groups of 4D and 3D supergravity theories are given by the conformal, $Conf(J)$, and quasiconformal groups $QConf(J)$, of $J$, respectively.
The quantization of the motion of fiducial particle on $\mathcal{M}_3^*$ leads to quantum mechanical wave functions that provide the basis of a unitary representation of the isometry group $G_3$ of $\mathcal{M}_3^*$. BPS black holes correspond to a special class of geodesics which lift holomorphically to the twistor space $Z_3$ of $\mathcal{M}_3^*$. Spherically symmetric stationary BPS black holes of $N=2$ MESGT's are described by holomorphic curves in $Z_3$. The relevant unitary representations of the 3D isometry groups $Q\text{Conf}(J)$ for BPS black holes are those induced by their holomorphic actions on the corresponding twistor spaces $Z_3$, which belong to quaternionic discrete series representations. For rank two quaternionic groups $SU(2,1)$ and $G_2(2)$ unitary representations induced by the geometric quasiconformal actions and their spherical vectors were studied in MG, Neitzke, Pavlyk, Pioline (2007).
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For $\nu = -(n_V + 2) + i \rho$ the quasiconformal action induces unitary representations that belong to the principle series. For special discrete values of $\nu$ the quasiconformal action leads to unitary representations belonging to the discrete series and their continuations. For the quaternionic real forms they belong to the quaternionic discrete series.
Minimal Unitary Representations and Quasiconformal
Groups

- Quantization of the quasiconformal realization of a non-compact Lie group leads directly to its minimal unitary representation \( \Rightarrow \) Unitary representation over the Hilbert space of square integrable functions of smallest number of variables possible.

- Minimal unitary dimension for \( E_{8(8)} \) is 29. Minimal unitary representation of \( E_{8(8)} \) from its geometric realization as a quasiconformal group action in 57 dimensions.

  GKN (2000)

\[
E_{8(8)} = 1_{-2} \oplus 56_{-1} \oplus E_{7(7)} + SO(1, 1) \oplus 56_{+1} \oplus 1_{+2}
\]

- 56 is a symplectic representation of \( E_{7(7)} \) \( \Rightarrow \) split 56 into 28 coordinates and 28 momenta. These 28 coordinates plus the singlet coordinate yield the minimal number of variables for \( E_8 \).
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- This construction has been extended to $E_{8(-24)}$ and generalized to all noncompact simple Lie groups MG, Pavlyk 2004.
QCG Approach to Minimal Unitary Representations

\[ g = g^{-2} \oplus g^{-1} \oplus (h \oplus \Delta) \oplus g^{+1} \oplus g^{+2} \]

\[ g = E \oplus E^\alpha \oplus (J^a + \Delta) \oplus F^\alpha \oplus F \]

\[ \Delta \text{ determines the 5-grading and } \alpha, \beta, \ldots = 1, 2, \ldots, 2n \]

\[ E = \frac{1}{2} y^2 \quad E^\alpha = y \xi^\alpha, \quad J^a = -\frac{1}{2} \lambda^a_{\alpha\beta} \xi^\alpha \xi^\beta \]

\[ F = \frac{1}{2} p^2 + \frac{\kappa l_4(\xi^\alpha)}{y^2}, \quad F^\alpha = [E^\alpha, F] \]

\[ [\xi^\alpha, \xi^\beta] = \Omega^{\alpha\beta} = -\Omega^{\beta\alpha}, \quad [y, p] = i \]

\[ l_4(\xi^\alpha) = S_{\alpha\beta\gamma\delta} \xi^\alpha \xi^\beta \xi^\gamma \xi^\delta \iff \text{quartic invariant of } h = g^0 / \Delta \]

minimal dimension \( = n + 1 \)
4D, $N = 2$ $\sigma$-models coupled to Supergravity in Harmonic Superspace and Minimal Unitary Representations

- In HSS the metric on a quaternionic target space is given by a quaternionic potential $\mathcal{L}^{(+4)}$. The action is

$$S = \int d\zeta^{(-4)} du \{ Q^+ \alpha^+ D^+ + Q^{+\alpha} - q_i^+ D^+ + q^{+i} + \mathcal{L}^{(+4)}(Q^+, q^+, u^-) \}$$

$\zeta$, $u_i^\pm$ are **analytic** superspace coordinates. Hypermultiplets $Q^+_{\alpha}(\zeta, u)$, $\alpha = 1, \ldots, 2n$ and supergravity compensators $q_i^+(\zeta, u)$, $(i = 1, 2)$ are analytic $N = 2$ superfields.
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- "Hamiltonian mechanics" with $D^{++}$ playing the role of time derivatives and $Q^+$ and $q^+$ corresponding to phase space coordinates under Poisson brackets

$$\{f, g\} = \frac{1}{2} \Omega^{\alpha\beta} \frac{\partial f}{\partial Q^{+\alpha}} \frac{\partial g}{\partial Q^{+\beta}} - \frac{1}{2} \epsilon^{ij} \frac{\partial f}{\partial q^{+i}} \frac{\partial g}{\partial q^{+j}},$$

Galperin, Ogievetsky 1993
Isometries are generated by Killing potentials \( K_A(Q^+, q^+, u^-) \) that obey the "conservation law"

\[
\partial^{++} K_A + \{ K_A, \mathcal{L}^{(4)} \} = 0
\]

\[
\mathcal{L}^{(4)} = \frac{P^{(4)}(Q^+)}{(q^+ u^-)^2}
\]

\[
P^{(4)}(Q^+) = \frac{1}{12} S_{\alpha\beta\gamma\delta} Q^{+\alpha} Q^{+\beta} Q^{+\gamma} Q^{+\delta}
\]

where \( S_{\alpha\beta\gamma\delta} \) is a completely symmetric invariant tensor of \( H \).

The Killing potentials that generate the isometry group \( G \) are given by

\[
\text{Sp}(2) : \quad K_{ij}^{++} = 2(q_i^+ q_j^+ - u_i^- u_j^- \mathcal{L}^{(4)}),
\]

\[
H : \quad K_a^{++} = t_{a\alpha\beta} Q^{+\alpha} Q^{+\beta},
\]

\[
G/H \times \text{Sp}(2) : \quad K_{i\alpha}^{++} = 2q_i^+ Q_{\alpha}^+ - u_i^- (q^+ u^-) \partial_{\alpha} \mathcal{L}^{(4)}
\]
<table>
<thead>
<tr>
<th>$\sigma$-model with Isometry Group $G$ in HSS</th>
<th>Minimal Unitary Representation of $G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w$</td>
<td>$y$</td>
</tr>
<tr>
<td>$p^{++}$</td>
<td>$p$</td>
</tr>
<tr>
<td>${ , , }$</td>
<td>$i[ , ]$</td>
</tr>
<tr>
<td>$Q^{+\alpha}$</td>
<td>$\xi^\alpha$</td>
</tr>
<tr>
<td>$P^{(+4)}(Q^+)$</td>
<td>$l_4(\xi)$</td>
</tr>
<tr>
<td>$K^{a^{++}} = t^a_{\alpha \beta} Q^{+\alpha} Q^{+\beta}$</td>
<td>$J^a = \chi^a_{\alpha \beta} \xi^\alpha \xi^\beta$</td>
</tr>
<tr>
<td>$T^{+++}<em>{\alpha} = K^{i^{++}}</em>{i \alpha} u^{+i}$</td>
<td>$F^\alpha$</td>
</tr>
<tr>
<td>$T^{+}<em>{\alpha} = K^{i^{++}}</em>{i \alpha} u^{-i}$</td>
<td>$E^\alpha$</td>
</tr>
<tr>
<td>$M^{i^{+++}}$</td>
<td>$F$</td>
</tr>
<tr>
<td>$M^0$</td>
<td>$E$</td>
</tr>
<tr>
<td>$M^{++}$</td>
<td>$\Delta$</td>
</tr>
</tbody>
</table>

**Table:** Above we give the correspondence between the harmonic superspace formulation of $N = 2$ sigma models coupled to supergravity and the operators that enter in the minimal unitary realizations of their isometry groups.

M. Günaydin, Miami 2010
The above mapping implies that the *fundamental spectra* of the quantum $N = 2$, quaternionic Kähler $\sigma$ models coupled to sugra in $d = 4$ must fit into the minimal unitary representations of their isometry groups.
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The \( N = 2 \), \( d = 4 \) MESGT’s under dimensional reduction lead to \( d = 3 \) supersymmetric \( \sigma \) models with quaternionic Kähler manifolds \( M_3 \) (C-map). After T-dualizing the three dimensional theory one can lift it back to four dimension, thereby obtaining an \( N = 2 \) sigma model coupled to supergravity that is in the mirror image of the original \( N = 2 \) MESGT.
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The above mapping shows that quantized solutions (spectrum) of a $4D, N = 2$ MESGT must fit into the minimal unitary representation of its three dimensional U-duality group and those representations obtained by tensoring of minimal representations.
The Kaluza-Klein spectrum of IIB supergravity on $AdS_5 \times S^5$ was first obtained via the oscillator method by simple tensoring of the CPT self-conjugate doubleton supermultiplet of $N = 8$ $AdS_5$ superalgebra $PSU(2, 2 | 4)$. The CPT self-conjugate doubleton supermultiplet of $PSU(2, 2 | 4)$ of $AdS_5 \times S^5$ solution of IIB supergravity does not have a Poincaré limit in five dimensions and decouples from the Kaluza-Klein spectrum as gauge modes and the field theory of CPT self-conjugate doubleton supermultiplet of $PSU(2, 2 | 4)$ lives on the boundary of $AdS_5$, which can be identified with 4D Minkowski space on which $SO(4, 2)$ acts as a conformal group, and the unique candidate for this theory is the four dimensional $N = 4$ super Yang-Mills theory that was known to be conformally invariant.
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The spectra of 11D supergravity over $AdS_4 \times S^7$ and $AdS_7 \times S^4$ were fitted into supermultiplets of the symmetry superalgebras $OSp(8 \mid 4, \mathbb{R})$ and $OSp(8^* \mid 4)$ constructed by oscillator methods. The entire Kaluza-Klein spectra over these two spaces were obtained by tensoring the singleton and doubleton supermultiplets of $OSp(8 \mid 4, \mathbb{R})$ and $OSp(8^* \mid 4)$, respectively.

The relevant singleton supermultiplet of $OSp(8 \mid 4, \mathbb{R})$ and doubleton supermultiplet of $OSp(8^* \mid 4)$ do not have a Poincaré limit in four and seven dimensions, respectively, and decouple from the respective spectra as gauge modes. Again it was proposed that field theories of the singleton and scalar doubleton supermultiplets live on the boundaries of $AdS_4$ and $AdS_7$ as superconformally invariant theories. MG, Warner (1984), MG, PvN, Warner (1984).
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Singletons of $Sp(4, \mathbb{R})$ are the remarkable representations of Dirac (1963). Subsequent important work of Fronsdal and collaborators on the singular nature of their Poincare limit and the massless representations of $SO(3, 2)$ obtained by tensoring the singletons.
Minimal Unitary Representations versus Singletons & Doubletons and Supersymmetry

MG, Pavlyk (2006), MG, Fernando (2009/10)

- The minimal unitary representations of symplectic groups $Sp(2n, \mathbb{R})$ are the singletons and their generators can be written as bilinears of bosonic oscillators since their quartic invariants vanish. Tensoring procedure becomes simple for the symplectic groups. Minreps of $OSp(2n|2m, \mathbb{R})$ are supersingletons
- Minimal unitary representation of $SU(2, 2) = SO(4, 2)$ over $L^2$ functions in 3 variables $\Leftrightarrow$ conformal scalar $= \text{scalar doubleton}$
- Minrep of $SU(2, 2)$ admits a one-parameter, $\zeta$, family of deformations corresponding to massless conformal fields in $d = 4$ with helicity $\zeta/2$.
- Minrep of $PSU(2, 2|4)$ is the 4D Yang-Mills supermultiplet (CPT-self-conjugate doubleton)
- Minrep of $SU(2, 2|N)$ admits a one-parameter family of deformations $\Leftrightarrow$ Higher spin doubletons studied by MG, Minic, Zagermann (1998).
Minrep of $SO^*(8) \cong SO(6,2)$ realized over the Hilbert space of functions of five variables and its deformations labeled by the spin $t$ of an $SU(2)$ subgroup correspond to massless conformal fields in six dimensions. Minimal unitary supermultiplet of $OSp(8^*|2N)$ admits deformations labeled uniquely by the spin $t$ of an $SU(2)$ subgroup of the little group $SO(4)$ of lightlike vectors in $6D$.

Minrep of $OSp(8^*|4)$ is the massless supermultiplet of $(2,0)$ conformal field theory that is dual to M-theory on $AdS_7 \times S^4$. 
OPEN PROBLEMS

- Decomposition of tensor products of minreps of U-duality groups into irreps. Of particular interest are $E_{8(8)}$ and $E_{8(-24)}$.
- Physical meaning of the states of minrep?
- For 3D U-duality groups of MESGTs with 8 real supersymmetries the fundamental spectrum corresponds to "super BPS" states, i.e. they preserve full susy!
- Construction of the full spectrum by tensoring of minreps and embedding into their quantum completion (M/Superstring theory).
- At the non-perturbative level one expects only the discrete subgroups of U-duality groups to be symmetries of M/superstring theorycompactified to various dimensions on tori or their orbifolds. How to extend the above results to the discrete subgroups? Automorphic representations?