Tunneling in classical mechanics

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To be discussed    arXiv: hep-th/1011.0121
The point of this talk:

Quantum mechanics is a strange animal...

But, by extending classical mechanics into the complex plane we can understand QM a bit better:

The quantum-classical correspondence extends beyond the classically allowed region and into the classically forbidden region where we can see the classical analog of *tunneling*!
Earlier work on extending quantum mechanics into the complex domain:  

**Dirac Hermiticity is too strong an axiom of quantum mechanics!**

\[ H = H^\dagger \]

\( \dagger \) means transpose + complex conjugate

- guarantees real energy and conserved probability
- but ... is a **mathematical** axiom and not a **physical** axiom of quantum mechanics
- idea: replace Dirac Hermiticity symmetry by weaker and more general **PT** symmetry (space-time reflection symmetry)
A class of $PT$-symmetric Hamiltonians:

$$H = p^2 + x^2 (ix)^\varepsilon \quad (\varepsilon \text{ real})$$
The spectrum of $H = p^2 + x^2 (i\hbar)^2$ is discrete, real, and positive, and parity symmetry is broken ($\epsilon > 0$).
The spectrum of $H = p^2 + x^2(\text{i}x)^6$ is discrete, real, and positive, and parity symmetry is broken if $\epsilon > 0$.
Some references ...


How to prove that the eigenvalues are real

The proof is difficult! It uses techniques from conformal field theory and statistical mechanics:

1. Bethe ansatz
2. Monodromy group
3. Baxter T-Q relation
4. Functional Determinants
\[ H = p^2 + x^2(ix)^\epsilon \quad (\epsilon \text{ real}) \]
The \textit{PT} Boundary is a phase transition – at the \textit{classical} level

(To be explained momentarily!)
OK, so the eigenvalues are real ... But is this quantum mechanics??

- Probabilistic interpretation??
- Hilbert space with a positive metric??
- Unitarity??
The Hamiltonian determines its own adjoint

\[[C, PT] = 0,\]
\[[C^2 = 1],\]
\[[C, H] = 0\]

Replace \(\dagger\) by \(CPT\)
Unitarity

With respect to the $CPT$ adjoint, the theory has UNITARY time evolution.

Norms are strictly positive! Probability is conserved!
Laboratory confirmation using table-top optics experiments!

Observing $PT$ symmetry using optical wave guides:

The observed $PT$ phase transition

**Figure 4:** Experimental observation of spontaneous passive $PT$-symmetry breaking. Output transmission of a passive $PT$ complex system as the loss in the lossy waveguide arm is increased. The transmission attains a minimum at 6 cm$^{-1}$. 

![Graph showing the relationship between transmission and loss](image-url)
Observation of parity-time symmetry in optics

Christian E. Rüter¹, Konstantinos G. Makris², Ramy El-Ganainy², Demetrios N. Christodoulides², Mordechai Segev³ and Detlef Kip¹*

One of the fundamental axioms of quantum mechanics is associated with the Hermiticity of physical observables¹. In the case of the Hamiltonian operator, this requirement not only implies real eigenenergies but also guarantees probability conservation. Interestingly, a wide class of non-Hermitian Hamiltonians can still show entirely real spectra. Among these are Hamiltonians respecting parity-time (PT) symmetry²-⁷. Even though the Hermiticity of quantum observables was never in doubt, such concepts have motivated discussions on several fronts in physics, including quantum field theories⁸, non-Hermitian Anderson models⁹ and open quantum systems¹⁰,¹¹, to mention a few. Although the impact of PT symmetry in these fields is still debated, it has been recently realized that optics can provide a fertile ground where PT-related notions can be implemented and experimentally investigated¹²-¹⁵. In this letter we report the first observation of the behaviour of a PT optical coupled system that judiciously involves a complex index potential. We observe both spontaneous PT symmetry breaking and power oscillations violating left-right symmetry. Our results may pave the way towards a new class of PT-synthetic materials with intriguing and unexpected properties that rely on non-reciprocal light propagation and tailored transverse energy flow.

(ε > εₜₜ), the spectrum ceases to be real and starts to involve imaginary eigenvalues. This signifies the onset of a spontaneous PT symmetry-breaking, that is, a ‘phase transition’ from the exact to broken-PT phase⁷,¹⁰.

In optics, several physical processes are known to obey equations that are formally equivalent to that of Schrödinger in quantum mechanics. Spatial diffraction and temporal dispersion are perhaps the most prominent examples. In this work we focus our attention on the spatial domain, for example optical beam propagation in PT-symmetric complex potentials. In fact, such PT ‘optical potentials’ can be realized through a judicious inclusion of index guiding and gain/loss regions⁷,¹²-¹⁴. Given that the complex refractive-index distribution \( n(x) = n_R(x) + i n_I(x) \) plays the role of an optical potential, we can then design a PT-symmetric system by satisfying the conditions \( n_R(x) = n_R(-x) \) and \( n_I(x) = -n_I(-x) \).

In other words, the refractive-index profile must be an even function of position \( x \) whereas the gain/loss distribution should be odd. Under these conditions, the electric-field envelope \( E \) of the optical beam is governed by the paraxial equation of diffraction¹³:

\[
\frac{\partial E}{\partial z} + \frac{1}{2k} \frac{\partial^2 E}{\partial x^2} + k_0 [n_R(x) + i n_I(x)] E = 0
\]
**Figure 2 | Experimental set-up.** An Ar⁺ laser beam (wavelength 514.5 nm) is coupled into the arms of the structure fabricated on a photorefractive LiNbO₃ substrate. An amplitude mask blocks the pump beam from entering channel 2, thus enabling two-wave mixing gain only in channel 1. A CCD camera is used to monitor both the intensity and phases at the output.
Figure 3 | Computed and experimentally measured response of a PT-symmetric coupled system. a, Numerical solution of the coupled equations (1) describing the PT-symmetric system. The left (right) panel shows the situation when light is coupled into channel 1 (2). Red dashed lines mark the symmetry-breaking threshold. Above threshold, light is predominantly guided in channel 1 experiencing gain, and the intensity of channels 1 and 2 depends solely on the magnitude of the gain. b, Experimentally measured (normalized) intensities at the output facet during the gain build-up as a function of time.
Another experiment...

“Enhanced magnetic resonance signal of spin-polarized Rb Atoms near surfaces of coated cells”
K. F. Zhao, M. Schaden, and Z. Wu
Physical Review A 81, 042903 (2010)
Abstract: We formulate a description of transport in a superconducting weak link in terms of the non-Hermitian quantum mechanics. We find that the applied electric field exceeding a certain critical value change the topological structure of the effective non-Hermitian Hamiltonian of the weak link in the Hilbert space causing the parity reflection – time reversal symmetry (PT-symmetry) breakdown. We derive the expression of the critical electric field and show that the PT-symmetry breakdown gives rise to the switching instability in the current-voltage characteristic of the weak link. Taking into account superconducting fluctuations we quantitatively describe the experimentally observed differential resistance of the weak link in the vicinity of the critical temperature.
Spontaneous Parity--Time Symmetry Breaking and Stability of Solitons in Bose-Einstein Condensates

Zhenya Yan, Bo Xiong, Wu-Ming Liu

(arXiv:1009.4023v1 [cond-mat.quant-gas], submitted on 21 Sep 2010)

Abstract: We report explicitly a novel family of exact PT-symmetric solitons and further study their spontaneous PT symmetry breaking, stabilities and collisions in Bose-Einstein condensates trapped in a PT-symmetric harmonic trap and a Hermite-Gaussian gain/loss potential. We observe the significant effects of mean-field interaction by modifying the threshold point of spontaneous PT symmetry breaking in Bose-Einstein condensates. Our scenario provides a promising approach to study PT-related universal behaviors in non-Hermitian quantum system based on the manipulation of gain/loss potential in Bose-Einstein condensates.
What exactly is this \textit{PT} phase transition?

Examining the CLASSICAL limit of \textit{PT} quantum mechanics provides an intuitive explanation of \textit{PT} symmetry…
Motion of particles is governed by Newton’s Law:

\[ F = ma \]

In freshman physics this motion is restricted to the REAL AXIS.
Harmonic oscillator: Particle on a spring

Back and forth motion on the real axis:

\[ H = p^2 + x^2 \quad (\epsilon = 0) \]
Harmonic oscillator

Motion in the complex plane:

\[ H = p^2 + x^2 \]

(\( \epsilon = 0 \))
\[ H = p^2 + ix^3 \quad (\epsilon = 1) \]
\[ \varepsilon = \pi - 2 \] Classical orbit visits 3 sheets of Riemann surface
Classical orbit that visits three sheets of the Riemann surface
\[ \epsilon = \pi - 2 \] visits 11 sheets of Riemann surface
Broken $PT$ symmetry – orbit not closed

$\varepsilon < 0$
Bohr-Sommerfeld
Quantization of a complex atom

\[ \int dx \ p = \left( n + \frac{1}{2} \right) \pi \]
Interesting recent developments...

(1) K. Jones-Smith and H. Mathur (Case Western): $PT$-symmetric Dirac equation and neutrino oscillations
(2) G. ‘t Hooft: cosmological models
(3) J. Moffat (Perimeter): cosmological constant
(4) M. de Kieviet (Heidelberg): experimental observations of $PT$-symmetric quantum brachistochrone
(5) P. Dorey, C. Dunning, R. Tateo: ODE-IM correspondence
(6) D. Masoero (Trieste): cubic $PT$ oscillator and Painleve I; quartic $PT$ oscillator and Painleve II
(7) S. Longhi (Milan): Bloch waves
(8) Classical $PT$-symmetric equations: KdV, Camassa-Holm, Sine-Gordon, Boussinesq, Lotka-Volterra, Euler’s; complex extension of chaos
(9) Complex quantum mechanics: Complex correspondence principle
(10) A. LeClair (Cornell): Generalization of spin and statistics
(11) H. Schomerous (Lancaster): PT quantum noise
(12) D. Christodoulides (Florida): Random $PT$ dimers
(13) ... And, QUANTUM TUNNELING --- This talk!
The point of this talk:

We can begin to understand quantum tunneling by studying complex classical mechanics!

In fact, in the complex plane classical mechanics strongly resembles quantum mechanics!
\[ V^{(4)}(x) = \frac{7}{2} x(x - 1) \left( x + \frac{191}{100} \right) \left( x - \frac{49}{20} \right) \]
Quantum probability densities:
Quantum probabilities:

\[
P_{\text{right,1}}^{\text{quant}} = 99.5933\%,
\]

\[
P_{\text{right,3}}^{\text{quant}} = 59.7584\%,
\]

\[
P_{\text{right,2}}^{\text{quant}} = 0.4316\%,
\]

\[
P_{\text{right,4}}^{\text{quant}} = 40.7689\%.
\]
Classical motion for real energy — **NO TUNNELING**: 
Classical motion when the energy is slightly complex — *TUNNELING*:
\[ P_{\text{right},1}^{\text{quant}} = 99.5933\%, \]
\[ P_{\text{right},3}^{\text{quant}} = 59.7584\%, \]
\[ P_{\text{right},2}^{\text{quant}} = 0.4316\%, \]
\[ P_{\text{right},4}^{\text{quant}} = 40.7689\% . \]

\[ P_{\text{right},1}^{\text{class}} = 55.4\%, \]
\[ P_{\text{right},3}^{\text{class}} = 54.3\%, \]
\[ P_{\text{right},2}^{\text{class}} = 55.0\%, \]
\[ P_{\text{right},4}^{\text{class}} = 55.0\%. \]
\[ V^{(6)}(x) = x^6 - 2x^5 - 4x^4 + 11x^3 - \frac{11}{4}x^2 - 13x \]
\[ P_{\text{right},0}^{\text{quant}} = 0.0391\%, \]
\[ P_{\text{right},2}^{\text{quant}} = 99.8651\%, \]
\[ P_{\text{right}}^{\text{quant}} = 99.9986\%, \]
\[ P_{\text{right},3}^{\text{quant}} = 78.7223\%. \]

\[ P_{\text{right},0}^{\text{class}} = 91.7\%, \]
\[ P_{\text{right},2}^{\text{class}} = 32.4\%, \]
\[ P_{\text{right},1}^{\text{class}} = 87.1\%, \]
\[ P_{\text{right},3}^{\text{class}} = 72.3\%. \]
Classical tunneling is like a quantum anomaly...

You get 0 unless you introduce a regulator that tends to 0.
This is like the Fourier sine series for $f(x) = 1$

Partial sum:  
\[ S_K(x) \equiv \frac{4}{\pi} \sum_{k=0}^{K} \frac{1}{2k+1} \sin[(2k+1)x] \]

While $S_K(x)$ at $x = 0$ provides no information about $f(0)$, the extrapolation procedure used above for classical tunneling probability can be used here to determine $f(0)$. We take twice as many terms in the partial sum as $x$ is halved and thereby circumvent the problem of nonuniform convergence (the Gibbs phenomenon). We evaluate $S_K(x)$ at $K = 100 \times 2^k$ and $x = 2^{-k}$ for $k = 0, 1, 2, 3, 4, 5$. The numerical values of $S_K(x)$ are

\[
S_{100} = 0.997776, \quad S_{200} = 0.996704, \quad S_{400} = 0.997293, \\
S_{800} = 0.997818, \quad S_{1600} = 0.998128, \quad S_{3200} = 0.998292.
\]

We easily infer from this sequence that $f(0) = 1$. 
Conclusion:

The quantum-classical correspondence extends beyond the classically allowed region and into the classically forbidden region where we can see the classical analog of *tunneling*!