$E_{10}$, $K(E_{10})$, and Unification

Hermann Nicolai  
MPI für Gravitationsphysik (AEI), Potsdam  
*Celebrating ten years of AdS/CFT*  
Lago Mar Resort, 13 - 18 December 2007

(mostly) based on work done in collaboration with:  
Thibault Damour, Axel Kleinschmidt and Marc Henneaux
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Main message of this talk:
Search for unification = search for symmetries
Most successful guiding principle of physics
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Main message of this talk:
Search for unification = search for symmetries
Most successful guiding principle of physics
... and perhaps also for quantum gravity...
The BKL Paradigm

Near a spacelike (cosmological) singularity, Einstein equations should simplify \( \Rightarrow \) BKL decoupling: \( \partial_x \ll \partial_t \)?

[BKL ≡ Belinskii, Khalatnikov, Lifshitz (1972)]
Near a spacelike (cosmological) singularity, Einstein equations should simplify ⇒ BKL decoupling: $\partial_x \ll \partial_t$?

[BKL = Belinskii, Khalatnikov, Lifshitz (1972)]
Near a spacelike (cosmological) singularity, Einstein equations should simplify $\Rightarrow$ BKL decoupling: $\partial_x \ll \partial_t$?

[BKL $\equiv$ Belinskii, Khalatnikov, Lifshitz (1972)]

Dimensional reduction to one (time) dimension $\rightarrow$ effective dynamics near singularity from gradient expansion? $\rightarrow$ billiards, chaotic oscillations, etc.
Another (old) paradigm

Cosmological evolution as ‘geodesic motion’ in the moduli space of 3-geometries [Wheeler, DeWitt,...]:

\[ M \equiv G^{(3)} = \frac{\{\text{spatial metrics } g_{ij}(x)\}}{\{\text{diffeomorphisms}\}} \]
Another (old) paradigm

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\[ \mathcal{M} \equiv G^{(3)} = \left\{ \text{spatial metrics } g_{ij}(x) \right\} / \left\{ \text{diffeomorphisms} \right\} \]

Can we understand and ‘simplify’ \( \mathcal{M} \) by means of an embedding into a group theoretical coset \( G/K(G) \)?
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The prototype example: moduli space of solutions of Einstein equations with two commuting Killing vectors

\[ \mathcal{M} = A_{1}^{(1)} / K(A_{1}^{(1)}) , \quad A_{1}^{(1)} \equiv SL(2, \mathbb{R})_{c.e.} = \text{Geroch group} \]
Another (old) paradigm

- Cosmological evolution as ‘geodesic motion’ in the moduli space of 3-geometries \([\text{Wheeler, DeWitt,\ldots}]\):

\[
\mathcal{M} \equiv G^{(3)} = \left\{ \text{spatial metrics } g_{ij}(x) \right\} / \{ \text{diffeomorphisms} \}
\]

- Can we understand and ‘simplify’ \(\mathcal{M}\) by means of an embedding into a group theoretical coset \(G/K(G)\)?

- The prototype example: moduli space of solutions of Einstein equations with two commuting Killing vectors

\[
\mathcal{M} = A_1^{(1)}/K(A_1^{(1)}) , \quad A_1^{(1)} \equiv SL(2,\mathbb{R})_{c.e.} = \text{Geroch group}
\]

- Unification of space-time, matter and gravitation: configuration space \(\mathcal{M}\) for quantum gravity should consistently incorporate matter degrees of freedom.

\(E_{10}\) and \(K(E_{10})\): re-inventing M theory? – p.3/16
### Hidden symmetries

Reduction of **SUGRA**\(_{11}\) to \(D = 11 - n\) [Cremmer, Julia (1979)]

<table>
<thead>
<tr>
<th>(n)</th>
<th>Scalar Coset (E_n/K(E_n))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(GL(1)/1)</td>
</tr>
<tr>
<td>2</td>
<td>(GL(2)/SO(2))</td>
</tr>
<tr>
<td>3</td>
<td>(SL(3) \times SL(2)/U(2))</td>
</tr>
<tr>
<td>4</td>
<td>(SL(5)/SO(5))</td>
</tr>
<tr>
<td>5</td>
<td>(SO(5,5)/SO(5) \times SO(5))</td>
</tr>
<tr>
<td>6</td>
<td>(E_6/USp(4))</td>
</tr>
<tr>
<td>7</td>
<td>(E_7/SU(8))</td>
</tr>
<tr>
<td>8</td>
<td>(E_8/(Spin(16)/\mathbb{Z}_2))</td>
</tr>
<tr>
<td>9</td>
<td>(E_9/K(E_9))</td>
</tr>
<tr>
<td>10</td>
<td>(E_{10}/K(E_{10}))</td>
</tr>
<tr>
<td>11</td>
<td>(E_{11}/K(E_{11}))</td>
</tr>
</tbody>
</table>

\(E_{10}\) and \(K(E_{10})\) : re-inventing M theory? – p.4/16
from dimensional reduction to $\mathbb{D} = 1$?

However: $L = L(g_{ij}(t))$, $A_{ijk}(t)$ is only invariant under $\text{GL}(10; \mathbb{R})_n$.

...but: Effective dynamics of diagonal metric degrees of freedom is governed by cosmological billiards in Weyl chamber of $E_{10}$!

[Damour, Henneaux, hep-th/0012172; DHN, hep-th/0212256]

Motivates basic conjecture: $M = E_{10} = K(E_{10})$.

Dynamics of supergravity (or some M theoretic extension) Null geodesic motion on $E_{10} = K(E_{10})$ coset space are equivalent! [DHN, hep-th/0207267]

SUGRA eqs. of motion + canonical constraints 1-component geodesic eqn. and coset constraints
$E_{10}$ from dimensional reduction to $D = 1$?
\[ E_{10} \text{ from dimensional reduction to } D = 1? \]

However: \[ \mathcal{L} = \mathcal{L}(g_{ij}(t), A_{ijk}(t)) \] is only invariant under \( GL(10, \mathbb{R}) \times T_{120} \) ... but:
$E_{10}$ from dimensional reduction to $D = 1$?

However: $\mathcal{L} = \mathcal{L}(g_{ij}(t), A_{ijk}(t))$ is only invariant under $GL(10, \mathbb{R}) \times T^{120} \ldots$ but:

Effective dynamics of diagonal metric degrees of freedom is governed by *cosmological billiards* in Weyl chamber of $E_{10}$!

[Damour, Henneaux, hep-th/0012172; DHN, hep-th/0212256]
$\Rightarrow E_{10}$ from dimensional reduction to $D = 1$?

However: $\mathcal{L} = \mathcal{L}(g_{ij}(t), A_{ijk}(t))$ is only invariant under $GL(10, \mathbb{R}) \times T_{120}$ ... but:

Effective dynamics of diagonal metric degrees of freedom is governed by *cosmological billiards* in Weyl chamber of $E_{10}$!

[Damour, Henneaux, hep-th/0012172; DHN, hep-th/0212256]

motivates **BASIC CONJECTURE:** $\mathcal{M} = E_{10}/K(E_{10})$

Dynamics of supergravity (or some M theoretic extension) and null geodesic motion on $E_{10}/K(E_{10})$ coset space are equivalent! [DHN, hep-th/0207267]

SUGRA eqs. of motion + canonical constraints $\Leftrightarrow$ $\infty$-component geodesic eqn. and coset constraints
Definition of $E_{10}$
**Definition of** \( E_{10} \)

\( E_{10} \) is the Kac–Moody group with Kac–Moody Lie algebra \( g \cong e_{10} \) of rank 10 defined via the Dynkin diagram.

![Dynkin diagram](image)

Cartan matrix \( A_{ij} \)

\( E_{10} \) and \( K(E_{10}) \): re-inventing M theory? – p.6/16
Definition of $E_{10}$

$E_{10}$ is the Kac–Moody group with Kac–Moody Lie algebra $\mathfrak{g} \equiv e_{10}$ of rank 10 defined via the Dynkin diagram

\[ 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \]

\[ A_{ij} \]

Cartan matrix

Chevalley–Serre presentation: Generators $h_i, e_i, f_i$ for $i = 1, \ldots, 10$ with relations

\[
\begin{align*}
[h_i, h_j] &= 0, \\
[h_i, e_j] &= A_{ij}e_j, \\
[h_i, f_j] &= -A_{ij}f_j, \\
(e_i)^{1-A_{ij}}e_j &= 0, \\
(f_i)^{1-A_{ij}}f_j &= 0.
\end{align*}
\]
Definition of $E_{10}$

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Chevalley–Serre presentation: Generators $h_i, e_i, f_i$ for $i = 1, \ldots, 10$ with relations

$$
[h_i, h_j] = 0, \quad [e_i, f_j] = \delta_{ij} h_i, \\
[h_i, e_j] = A_{ij} e_j, \quad [h_i, f_j] = -A_{ij} f_j, \\
(ad e_i)^{1-A_{ij}} e_j = 0, \quad (ad f_i)^{1-A_{ij}} f_j = 0.
$$

$h_i$ span Cartan subalgebra $\mathfrak{h}$; $e_i$ and $f_i$: positive and negative simple root generators

$E_{10}$ and $K(E_{10})$: re-inventing M theory? – p.6/16
Key Properties
Key Properties

Root space decomposition: \( \alpha \in Q(E_{10}) = \Pi_{1,9} \)

\[ g_\alpha = \{ x \in g : [h, x] = \alpha(h)x \quad \text{for} \ h \in h \} \]

Real roots \( (\alpha^2 = 2) \) and imaginary roots \( (\alpha^2 \leq 0) \)
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- Weyl group: $W^+(E_{10}) = \text{PSL}_2(\mathbb{O}_\mathbb{Z})$ [KN+Feingold (2007)]
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- Weyl group: $W^+(E_{10}) = \text{PSL}_2(\mathbb{O}_\mathbb{Z})$ [KN+Feingold (2007)]

- Invariant bilinear form $\rightarrow$ Action Principle

  $$\langle h_i | h_j \rangle = A_{ij}, \quad \langle e_i | f_j \rangle = \delta_{ij}, \quad \langle [x, y] | z \rangle = \langle x | [y, z] \rangle.$$  

  [No other polynomial Casimir for dim $\mathfrak{g} = \infty \rightarrow$ action is (essentially) unique!]
Key Properties

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  [No other polynomial Casimir for \( \dim g = \infty \) → action is (essentially) unique!]

- Triangular decomposition
  \[ g = e_{10} = n_- \oplus h \oplus n_+ \]
  \( \text{with } n_\pm := \bigoplus_{\alpha \geq 0} g_{\alpha} \)

\( E_{10} \) and \( K(E_{10}) \): re-inventing M theory? – p.7/16
Compact subalgebra $K(e_{10})$

Chevalley involution on $e_{10}$ is defined by

$$(e_i) = f_i; (f_i) = e_i; (h_i) = h_i$$

and extends to all of $e_{10}$ by

$$(x; y) = [x; y].$$

Fixed point set $K_{e_{10}} = x^2 e_{10}$ is a subalgebra of $e_{10}$, called the compact subalgebra.

However, $K_{e_{10}}$ is not a Kac-Moody algebra [KN, hep-th/0506238].
Compact subalgebra $K(e_{10})$

Chevalley involution $\omega$ on $e_{10}$ is defined by

$$\omega(e_i) = -f_i, \quad \omega(f_i) = -e_i, \quad \omega(h_i) = -h_i$$

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Fixed point set

$$\mathfrak{e}_{10} \equiv K(e_{10}) = \{ x \in e_{10} : \omega(x) \equiv -x^T = x \}$$

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Fixed point set

$$k_{10} \equiv K(e_{10}) = \{x \in e_{10} : \omega(x) \equiv -x^T = x\}$$

is a subalgebra of $e_{10}$, called the compact subalgebra.

→ generalizes compact subalgebra of finite dimensional Lie algebras (in split real form; e.g. $so(n) \subset gl(n)$)
Compact subalgebra $K(e_{10})$

Chevalley involution $\omega$ on $e_{10}$ is defined by

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Level decomposition: $A_9 \subset \epsilon_{10}$
Level decomposition: $A_9 \subset \mathfrak{e}_{10}$

These are just the representations corresponding to the bosonic fields of $D=11$ SUGRA and their magnetic duals. At level $\ell = 3$:

For more representations, see: Fischbacher, N. hep-th/0301017
Level decomposition: \( A_9 \subset e_{10} \)

\[
\begin{array}{|c|c|c|}
\hline
\ell & A_9 \text{ module} & \text{Tensor} \\
\hline
0 & [1000000001] \oplus [0000000000] & K_{a}^{\ b} \\
1 & [000000100] & E^{abc} \\
2 & [0001000000] & E^{a_1\ldots a_6} \\
3 & [010000001] & E^{a_1\ldots a_8 | a_9} \\
\hline
\end{array}
\]

\[ \mathfrak{sl}(10) \equiv A_9 \subset e_{10} \]
These are just the representations corresponding to the bosonic fields of $D = 11$ SUGRA and their magnetic duals.

At level $\ell = 3$: dual graviton $h_{a_1\ldots a_8 | a_9}$ (with $h_{[a_1\ldots a_8 | a_9]} = 0$)

[For more representations, see: Fischbacher, N. hep-th/0301017]
Versatility of $E_{10}$ ($\& E_{11}$)
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The one-dimensional $E_{10}$ $\sigma$-model unifies
Versatility of $E_{10}$ (& $E_{11}$)

The one-dimensional $E_{10}$ $\sigma$-model unifies

$\mathfrak{sl}(10) \subseteq e_{10}$

$D = 11$ SUGRA

[DHN; West 2002]
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$\mathfrak{sl}(10) \subseteq \mathfrak{e}_{10}$

$D = 11$ SUGRA

[DHN; West 2002]

$\mathfrak{so}(9, 9) \subseteq \mathfrak{e}_{10}$

mIIA $D = 10$ SUGRA

[Kleinschmidt, Schnakenburg, West 2003]
[Kleinschmidt, N. 2004]
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$mIIA \ D = 10$ SUGRA

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$\mathfrak{sl}(9) \oplus \mathfrak{sl}(2) \subseteq e_{10}$

$IIB \ D = 10$ SUGRA

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IIB $D = 10$ SUGRA

[Kleinschmidt, Schnakenburg, West 2003]

[Kleinschmidt, N. 2004]

These are the (maximal) low energy theories of the ‘M-theory diagram’, now all part of a single model.
Dynamics: bosonic Lagrangian

Decompose Cartan form for $V(t)^2 E_{10} = K(E_{10})$

$\mathbb{V}_t^1(V(t)) = Q(t) + P(t)$

$Q_2 e^{10} k^1_0$

$P_2 e^{10} k^1_0$

essentially unique coset Lagrangian ($n(t)$ = lapse)

$L = \frac{1}{2} n h P^j P_j$

invariant under local $K(E_{10})$ and global $E_{10}$:

$V(t) = k^1_0 P_k$

Equations of motion:

null geodesic on $E_{10} = K(E_{10})$

$n @ t (n P^j P_j) = Q; P^j P_j = 0$
Dynamics: bosonic Lagrangian

Decompose Cartan form for $\mathcal{V}(t) \in E_{10}/K(E_{10})$

$$\partial_t \mathcal{V} \mathcal{V}^{-1}(t) = Q(t) + \mathcal{P}(t) \quad , \quad Q \in k_{10} \; , \; \mathcal{P} \in e_{10} \otimes k_{10}$$
Dynamics: bosonic Lagrangian

Decompose Cartan form for $\mathcal{V}(t) \in E_{10}/K(E_{10})$

$$\partial_t \mathcal{V} \mathcal{V}^{-1}(t) = Q(t) + \mathcal{P}(t), \quad Q \in \mathfrak{k}_{10}, \; \mathcal{P} \in \mathfrak{e}_{10} \ominus \mathfrak{k}_{10}$$

$\Rightarrow$ essentially unique coset Lagrangian $(n(t) = \text{lapse})$

$$\mathcal{L} = \frac{1}{2n} \langle \mathcal{P} | \mathcal{P} \rangle.$$
**Dynamics: bosonic Lagrangian**

Decompose Cartan form for $\mathcal{V}(t) \in E_{10}/K(E_{10})$

$$\partial_t \mathcal{V} \mathcal{V}^{-1}(t) = Q(t) + \mathcal{P}(t) \quad , \quad Q \in \mathfrak{k}_{10} \quad , \quad \mathcal{P} \in \mathfrak{e}_{10} \oplus \mathfrak{k}_{10}$$

$\Rightarrow$ essentially unique coset Lagrangian ($n(t) =$ lapse)

$$\mathcal{L} = \frac{1}{2n} \langle \mathcal{P} | \mathcal{P} \rangle .$$

invariant under **local** $K(E_{10})$ and **global** $E_{10}$:

$$\mathcal{V}(t) \rightarrow k(t) \mathcal{V}(t) g \ \Rightarrow \ \mathcal{P} \rightarrow k \mathcal{P} k^{-1} , \ \mathcal{Q} \rightarrow k \mathcal{Q} k^{-1} + \partial_t kk^{-1}$$
Dynamics: bosonic Lagrangian

Decompose Cartan form for $\mathcal{V}(t) \in E_{10}/K(E_{10})$

$$\partial_t \mathcal{V}^{-1}(t) = Q(t) + P(t) \quad , \quad Q \in \mathfrak{k}_{10} \ , \ P \in \mathfrak{e}_{10} \ominus \mathfrak{k}_{10}$$

$\Rightarrow$ essentially unique coset Lagrangian ($n(t)$ = lapse)

$$\mathcal{L} = \frac{1}{2n} \langle P | P \rangle.$$  

invariant under local $K(E_{10})$ and global $E_{10}$:

$$\mathcal{V}(t) \rightarrow k(t) \mathcal{V}(t)g \Rightarrow \quad P \rightarrow kPK^{-1} \quad , \quad Q \rightarrow kQk^{-1} + \partial_t kk^{-1}$$

Equations of motion: null geodesic on $E_{10}/K(E_{10})$

$$n \partial_t (n^{-1}P) = [Q, P], \quad \langle P | P \rangle = 0.$$
Example: $A_9 \subset E_{10}$
Example: $A_9 \subset E_{10}$

With $\partial_t \mathcal{V} \mathcal{V}^{-1} = \sum_{\ell \geq 0} P^{(\ell)} \ast E^{(\ell)}$ (schematically) and truncation $P^{(\ell)} = 0$ for $\ell > 3$ $\Rightarrow$

Equations of motion up to $\ell = 3$ ($a, b = 1, \ldots, 10$) [DHN; DN, hep-th/0410245]

\[
 nD^{(0)}(n^{-1}P^{(0)}_{ab}) = -\frac{1}{4} (P^{(1)}_{acd} P^{(1)}_{bcd} - \frac{1}{9} \delta_{ab} P^{(1)}_{cde} P^{(1)}_{cde}) \\
 - \frac{1}{2 \cdot 5!} (P^{(2)}_{ac1 \ldots c5} P^{(2)}_{bc1 \ldots c5} - \frac{1}{9} \delta_{ab} P^{(2)}_{c1 \ldots c6} P^{(2)}_{c1 \ldots c6}) \\
 + \frac{4}{9!} (P^{(3)}_{ac1 \ldots c7|c8} P^{(3)}_{bc1 \ldots c7|c8} + \frac{1}{8} P^{(3)}_{c1 \ldots c8|a} P^{(3)}_{c1 \ldots c8|b} \\
 - \frac{1}{8} \delta_{ab} P^{(3)}_{c1 \ldots c8|c9} P^{(3)}_{c1 \ldots c8|c9})
\]

\[
 nD^{(0)}(n^{-1}P^{(1)}_{abc}) = -\frac{1}{6} P^{(2)}_{abcdef} P^{(1)}_{def} + \frac{1}{3 \cdot 5!} P^{(3)}_{abcd1 \ldots d5|d6} P^{(2)}_{d1 \ldots d6}
\]

\[
 nD^{(0)}(n^{-1}P^{(2)}_{a1 \ldots a6}) = \frac{1}{6} P^{(3)}_{a1 \ldots a6cde} P^{(1)}_{cde}
\]

\[
 nD^{(0)}(n^{-1}P^{(3)}_{a1 \ldots a8|a9}) = 0 \quad \text{(with } P^{(3)}_{[a1 \ldots a8|a9]} = 0).\]

This is a consistent truncation of $E_{10}/K(E_{10})$ coset dynamics: solutions of truncated theory are also solutions of the full theory.
Correspondence with SUGRA$_{11}$
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Bosonic $D = 11$ supergravity equations [Cremmer, Julia, Scherk 1978]

\[ \mathcal{E}_{AB} \equiv R_{AB} - \frac{1}{3} F_{ACDE} F_{B}^{CDE} + \frac{1}{36} \eta_{AB} F_{CDE} F_{CDE}^{CDE} = 0 \]

\[ \mathcal{M}^{BCD} \equiv D_A F^{ABCD} + \frac{1}{576} \epsilon^{BCDE_1...E_8} F_{E_1...E_4} F_{E_5...E_8} = 0 \]

and Bianchi identities:  $D_{[A} F_{BCDE]} = R_{[AB} C]D = 0$
Correspondence with SUGRA\textsubscript{11}

Bosonic $D = 11$ supergravity equations [Cremmer, Julia, Scherk 1978]

\begin{align*}
\mathcal{E}_{AB} &\equiv R_{AB} - \frac{1}{3} F_{ACDE} F_{B}^{CDE} + \frac{1}{36} \eta_{AB} F_{CDE} F^{CDE} = 0 \\
\mathcal{M}^{BCD} &\equiv D_{A} F^{ABCD} + \frac{1}{576} e^{BCDE_{1}...E_{8}} F_{E_{1}...E_{4}} F_{E_{5}...E_{8}} = 0
\end{align*}

and Bianchi identities: $D_{[A} F_{BCDE]} = R_{[AB C]D} = 0$

Consider gauge fixed (à la ADM) equations at some fixed spatial point $x_{0}$:

- keeping all temporal and first order spatial derivatives at $x_{0}$

- zero-shift gauge: $E_{M}^{A} = \left( \begin{array}{c|c} N & 0 \\ \hline 0 & e_{m}^{a} \end{array} \right)$ and Coulomb gauge: $A_{tmn} = 0$

- Anholonomy coefficients $[\partial_{b}, \partial_{c}] = \tilde{\Omega}_{bc|a} \partial_{a}$ chosen traceless (in some neighborhood of $x_{0}$) by exploiting spatial Lorentz group, i.e. $\Lambda_{ab} = \Lambda_{ab}(t, x)$ [???]

- Thus the standard ADM procedure leads to usual split into:

  - Dynamical equations: $\mathcal{E}_{ab} = \mathcal{M}_{abc} = D_{[0} F_{bcde]} = R_{[0a b]c} = 0$
  
  - Canonical constraints: $\mathcal{E}_{00} = \mathcal{E}_{0a} = \mathcal{M}_{0ab} = D_{[a} F_{bcde]} = R_{[ab c]d} = 0$

$E_{10}$ and $K(E_{10})$: re-inventing M theory? – p.13/16
Correspondence with SUGRA

Bosonic $D = 11$ supergravity equations [Cremmer, Julia, Scherk 1978]

$$\mathcal{E}_{AB} \equiv R_{AB} - \frac{1}{3} F_{ACDE} F_B^{CDE} + \frac{1}{36} \eta_{AB} F_{CDE} F^{CDE} = 0$$

$$\mathcal{M}^{BCD} \equiv D_A F^{ABCD} + \frac{1}{576} \epsilon^{BCDE_1...E_8} F_{E_1...E_4} F_{E_5...E_8} = 0$$

and Bianchi identities: $D_{[A} F_{BCDE]} = R_{[AB C]D} = 0$

Then with the identification $n = N e^{-1}$ and (r.h.s. always at fixed spatial point $x = x_0$)

$$D^{(0)} P^{(0)}_{ab} = R_{ab}^{\text{time derivatives}}$$

$$P^{(1)}_{abc} = N F_{0abc}$$

$$P^{(2)}_{a_1...a_6} = -\frac{1}{4!} N e_{a_1...a_6 b_1...b_4} F_{b_1...b_4}$$

$$P^{(3)}_{a_1...a_8|a_9} = \frac{3}{2} N e_{a_1...a_8 bc} \tilde{\Omega}_{bc|a_9}$$

the two sets of dynamical equations coincide! (recall $P^{(3)}_{[a_1...a_8|a_9]} = 0 \iff \tilde{\Omega}_{ab|b} = 0$)

Dynamical equations for $\text{mIIA}$ and $\text{IIB}$ similarly from level decompositions w.r.t. finite dimensional subgroups $D_9 \equiv SO(9, 9) \subset E_{10}$ and $A_8 \times A_1 \equiv SL(9) \times SL(2) \subset E_{10}$.

$E_{10}$ and $K(E_{10})$: re-inventing M theory? – p.13/16
Constraints: an intriguing link
Conserved $E_{10}$ current $J = n \mathcal{V} \mathcal{V}^{-1}$ (≡ Noether charge associated with global $E_{10}$):

$$J = \frac{1}{9!} J^{m_0|m_1...m_8} F_{m_0|m_1...m_8} + \frac{1}{6!} J^{m_1...m_6} F_{m_1...m_6} + \frac{1}{3!} J^{mn} F_{mn}$$

$$+ J_{(0)m} K^{m} n + \frac{1}{3!} J_{(1) mnp} E^{mnp} + \frac{1}{6!} J_{(2) m_1...m_6} E^{m_1...m_6} + \ldots$$
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Consider Sugawara-like ($\propto \mathcal{J} \otimes \mathcal{J}$) expressions [DKN, hep-th 0709.2691]

$$\mathcal{L}_{m_1 \ldots m_10; n_0|n_1 \ldots n_7}^{(-6)} = J_{n_0|m_1 \ldots m_8}^{(-3)} J_{m_9|m_10 n_1 \ldots n_7}^{(-3)}$$

$$\mathcal{L}_{m_1 \ldots m_10; n_1 \ldots n_5}^{(-5)} = J_{n_1 \ldots n_4 m_1 m_2}^{(-2)} J_{m_3 \ldots m_10}^{(-3)}$$

$$\mathcal{L}_{m_1 \ldots m_10; n_1 n_2}^{(-4)} = \frac{21}{5} J_{n_1 m_1 \ldots m_5}^{(-2)} J_{n_2 m_6 \ldots m_10}^{(-2)} + J_{n_1 m_1 m_2}^{(-1)} J_{n_2 | m_3 \ldots m_10}^{(-3)}$$
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\[
\mathcal{J} = \frac{1}{9!} J_{(-3)}^m n_{0} | m_{1} \ldots m_{10} \rangle \langle n_{1} \ldots n_{7} |
\]

\[
+ \frac{1}{6!} J_{(-2)}^m n_{0} | m_{1} \ldots m_{6} \rangle \langle n_{1} \ldots n_{6} |
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\[
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\]

\[
\mathcal{L}_{(-5)}^{m_{1} \ldots m_{10} ; n_{1} \ldots n_{5}} = J_{(-2)}^{n_{1} \ldots n_{4} m_{1} m_{2}} J_{(-3)}^{m_{3} \ldots m_{10}}
\]

\[
\mathcal{L}_{(-4)}^{m_{1} \ldots m_{10} ; n_{1} n_{2}} = \frac{21}{5} J_{(-2)}^{m_{1} \ldots m_{5}} J_{(-2)}^{n_{2} m_{6} \ldots m_{10}} + J_{(-1)}^{m_{1} m_{2} n_{1} \ldots n_{7}} J_{(-3)}^{n_{2} | m_{3} \ldots m_{10}}
\]

(with appropriate antisymmetrizations) to re-express canonical constraints:

\[
\mathcal{L}_{(-6)}^{m_{1} \ldots m_{10} ; n_{0} | n_{1} \ldots n_{7}} \propto \epsilon^{m_{1} \ldots m_{10}} \epsilon^{n_{1} \ldots n_{7} p q r} R_{pq r n_{0}} \quad \text{Bianchi (I)}
\]

\[
\mathcal{L}_{(-5)}^{m_{1} \ldots m_{10} ; n_{1} \ldots n_{5}} \propto \epsilon^{m_{1} \ldots m_{10}} \epsilon^{n_{1} \ldots n_{5} p_{1} \ldots p_{5}} D_{p_{1} F_{p_{2} \ldots p_{5}}} \quad \text{Bianchi (II)}
\]

\[
\mathcal{L}_{(-4)}^{m_{1} \ldots m_{10} ; n_{1} n_{2}} \propto \epsilon^{m_{1} \ldots m_{10}} \mathcal{M}^{0 n_{1} n_{2}} \quad \text{Gauss constraint}
\]

\[
\mathcal{L}_{(-3)}^{m_{1} \ldots m_{9}} \propto \epsilon^{m_{1} \ldots m_{9}} \mathcal{E}_{0 n} \quad \text{Momentum constraint}
\]
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  - unique (bosonic) action, Chern–Simons couplings;
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$E_{10}$ and $K(E_{10})$: re-inventing M theory? – p.15/16
Outlook

New mechanism for (de-)emergence of space-time?

Space from Lie algebra

Time `operationally' from Wheeler-DeWitt equation

General covariance as an emergent property?

New perspectives for background independence?

Quantization: wave function of the universe as a modular form over $E_{10}(\mathbb{Z})$?

$[\text{Ganor, hep-th/9903110}].$

Any relation to zero tension limit of string theory?

Further exploration of these links could lead to important advances in physics and mathematics.

Thank you for your attention.
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