Remodeling the B-model: Implications for large $N$ duality

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Never again will a single story be told as if it is the only one.

- John Berger
We propose a new, **complete**, B-model formalism to compute open and closed topological string amplitudes on mirrors of toric Calabi-Yau threefolds.

The formalism is **nonperturbative** in the moduli, hence can be used to do precise comparisons with the perturbative gauge theory obtained through large $N$ duality.

**References:**

- V.B., A. Klemm, M. Mariño and S. Pasquetti, 0709.1453
- V.B., A. Klemm, M. Mariño, S. Pasquetti and M. Weiss, work in progress
- M. Mariño, hep-th/0612127
- B. Eynard and N. Orantin, math-ph/0702045
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Why is that interesting?

- Various mathematical applications:
  - (Open) orbifold Gromov-Witten theory [BKMP]
  - Relation with matrix models and nonperturbative approach to topological strings [Mariño,EMO,MSW]
  - Hodge integrals, Hurwitz numbers, . . . [BM]

- Topological strings/Chern-Simons theory provides a toy model of AdS/CFT
  - Using our formalism, we can test this duality in very much detail in the perturbative regime of Chern-Simons theory [BKMP]
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2. New approach to B-model topological strings

3. Implications for large $N$ duality
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Gopakumar-Vafa correspondence

Start with the geometric transition:

\[ \text{local } \mathbb{P}^1 \rightarrow \text{conifold} \leftarrow T^* S^3 \]

Consider A-model topological string theory on both sides:

**LHS**: Closed topological strings on local $\mathbb{P}^1$
- defined as a perturbative expansion in $e^{-t}$ at large radius $t \to \infty$ ($t$ is the Kähler parameter of $\mathbb{P}^1$)

**RHS**: Open topological strings on $T^* S^3$ with $N$ D-branes wrapped around $S^3$
- Worldvolume gauge theory is $U(N)$ Chern-Simons theory on $S^3$ [Witten]

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The main statement

Strings on local $\mathbb{P}^1 \leftrightarrow U(N)$ Chern-Simons on $S^3$

- Kähler parameter $t \leftrightarrow 't$ Hooft parameter of CS
- Perturbative CS in $t \leftrightarrow$ expansion of topological strings at small radius $t \rightarrow 0 \ldots$ we don’t know how to do that in general!
- Topological statement of weak/strong duality

- We can study CS non-perturbatively [Witten], and compare with topological strings at large radius [GV]
- We can try to define A-model at small radius and compare with perturbative CS (goal of this talk)
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Consider probe D-branes in local $\mathbb{P}^1$, which are pushed through the geometric transition:

What we get is:

**Open version**

Open strings on local $\mathbb{P}^1 \leftrightarrow$ Wilson loops in CS on $S^3$ [OV]
Generalization: quotienting Gopakumar-Vafa

Quotient both sides of the transition by \( \mathbb{Z}_2 \), and resolve the fixed point:

\[
\text{local } \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \text{singular } \leftarrow T^*(S^3/\mathbb{Z}_2)
\]

We get:

Strings on local \( \mathbb{P}^1 \times \mathbb{P}^1 \leftrightarrow U(N) \text{ CS on } S^3/\mathbb{Z}_2 \) [Mariño,AKMV]

- Can also consider \( \mathbb{Z}_p \) quotients
  \( \Rightarrow A_{p-1} \text{ fibration over } \mathbb{P}^1 \leftrightarrow \text{lens space } L(p, 1) \)
- Open strings \( \leftrightarrow \) Wilson loops in CS theory
- Interesting to study more general threefolds than just local \( \mathbb{P}^1 \rightarrow \text{toric Calabi-Yau threefolds} \)
Difficult to study A-model at small radius directly

Use mirror symmetry to map the problem to the B-model, which is nonperturbative in the moduli, hence can be expanded at small radius

We want a complete formalism to compute B-model amplitudes nonpertubatively in the moduli, for the mirrors threefolds to toric Calabi-Yau threefolds
Outline

1. Large $N$ duality in topological strings

2. New approach to B-model topological strings

3. Implications for large $N$ duality
Mirror pairs \((X, Y)\) of Calabi-Yau threefolds

1. \(X\): toric Calabi-Yau threefold (noncompact)
   \(\rightarrow\) Examples: local \(\mathbb{P}^1\), local \(\mathbb{P}^1 \times \mathbb{P}^1\), local \(\mathbb{P}^2\), . . .

2. \(Y\): \(\{ww' = H(x, y)\} \subset \mathbb{C}^2 \times (\mathbb{C}^*)^2\)
   \(\rightarrow\) conic fibration over \(\mathbb{C}^* \times \mathbb{C}^*\), such that the conic fiber degenerates to 2 lines over a Riemann surface (the mirror curve)

\[\Sigma : \{H(x, y) = 0\} \subset (\mathbb{C}^*)^2.\]

\(\Sigma\) encodes the mirror geometry \(Y\).
Open mirror symmetry

1. noncompact probe brane in $X$
   $\rightarrow$ dual to Wilson loops in corresponding CS theory

2. brane wrapping one of the two lines over $\Sigma$
   $\rightarrow$ moduli space of the brane is the mirror curve $\Sigma$

So we want to study:

Open B-model topological strings on $Y$ of the form above with branes wrapping one of the two lines over $\Sigma$

- The open B-model amplitudes should live on the mirror curve $\Sigma$, since this is the open/closed moduli space (the moduli space of vacua of the theory)
- We want to build a set of open and closed amplitudes living on a Riemann surface $\Sigma$
Suppose that $\Sigma$ is the spectral curve of a matrix model.

- Loop equations of the matrix model $\Rightarrow$ recursive solution for the free energies $F_g$ and the correlation functions $W_k^{(g)}$ of the matrix model [EO]
- $F_g$: functions on $\Sigma$
- $W_k^{(g)}$: meromorphic differentials on $\Sigma$

The recursion solution is entirely geometric on $\Sigma$
$\Rightarrow$ can be defined for any $\Sigma$, whether it is the spectral curve of a matrix model or not
$\Rightarrow$ what do the $F_g$ and $W_k^{(g)}$ mean?
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Our claim

When $\Sigma$ is a mirror curve, the $F_g$ are the genus $g$ closed B-model amplitudes, while the $W^{(g)}_k$ are the genus $g$, $k$ holes open B-model amplitudes.

Why?

- In [Mariño, BKMP], we checked many many examples, and it works :-)
- In [DV], a sketch of a proof of the recursion from Kodaira-Spencer theory is proposed
- From large $N$ duality, for some toric geometries there are CS duals, which can be expressed as matrix models [Mariño], with spectral curves given by the mirror curve

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Start with a $\Sigma : \{H(x, y) = 0\} \in \mathbb{C}^* \times \mathbb{C}^*$. 

1. **the ramification points** $q_i \in \Sigma$ of the $x$-projection $\Sigma \to \mathbb{P}^1$:

$$\frac{\partial H}{\partial y}(q_i) = 0$$

- near $q_i$, there are 2 points $q, \bar{q}$ with $x(q) = x(\bar{q})$

2. **the disk amplitude** (different from EO because of $\mathbb{C}^* \times \mathbb{C}^*$)

$$\tilde{\mathcal{W}}_1^{(0)}(x) = \omega(x) = \log y(x) \frac{dx}{x}$$

3. **the annulus amplitude**

$$\tilde{\mathcal{W}}_2^{(0)}(x_1, x_2) = B(x_1, x_2) - \frac{dx_1 dx_2}{(x_1 - x_2)^2},$$

where $B(x_1, x_2)$ is the Bergman kernel of $\Sigma$. 
The Bergman kernel of $\Sigma$ is the unique meromorphic differential $B(x_1, x_2)$ with a double pole at $x_1 = x_2$ with no residue and no other pole, and normalized such that

$$\oint_{A_i} B(x_1, x_2) = 0,$$

where $(A_i, B^I)$ is a canonical basis of cycles for $\Sigma$.

- example: if $\Sigma$ has genus 0, in local coordinate $z$:

$$B(p, q) = \frac{dz(p)dz(q)}{(z(p) - z(q))^2}$$

- define also locally near a $q_i$:

$$dE_q(p) = \frac{1}{2} \int_{q}^{\bar{q}} B(\xi, p)$$
The recursion: First step

Fix:

\[ W_1^{(0)}(p_1) = 0, \quad W_2^{(0)}(p_1, p_2) = B(p_1, p_2). \]

The other differentials are generated recursively by

\[ W_g(p, p_1 \cdots, p_h) = \sum_{q_i} \text{Res}_{q=q_i} \frac{dE_q(p)}{\omega(q)} \left[ W_{g-1}(q, \bar{q}, p_1, \cdots, p_h) \right. \]

\[ + \sum_{l=0}^{g} \sum_{J \subset H} W_{g-l}(q, p_J) W_{l}(\bar{q}, p_{H \setminus J}) \]
Let $\phi(p)$ be an arbitrary anti-derivative of $\omega(p)$ (i.e. $d\phi = \Phi$). Then

$$F_g = \frac{1}{2 - 2g} \sum_{q_i} \text{Res}_{q=q_i} \phi(q) W_1^{(g)}(q)$$

Our claim

- $F_g$: genus $g$ closed B-model amplitudes
- $A_h^{(g)} = \int W_h^{(g)}$: genus $g$, $h$ hole open B-model amplitudes
Summary so far

- **Complete** formalism to generate open/closed B-model amplitudes (reproduces topological vertex — A-model — results at large radius)
- Only need the *disk* and the *annulus* amplitudes, generate everything else recursively
- **Gluing** formalism for topological string amplitudes

**Main point for this talk**

Formalism is valid at *any point* in moduli space, hence can be used to compute amplitudes at the point mirror to the small radius point.
Outline

1. Large $N$ duality in topological strings
2. New approach to B-model topological strings
3. Implications for large $N$ duality
Comparing with perturbative Chern-Simons theory

Highly non-trivial example:

Open strings on local $\mathbb{P}^1 \times \mathbb{P}^1 \leftrightarrow$ Wilson loops in CS on $S^3/\mathbb{Z}_2$

1. Open strings on local $\mathbb{P}^1 \times \mathbb{P}^1$ at small radius
   → go to the mirror B-model and use our formalism

2. Perturbative expansion of expectation values of Wilson loops in CS on $S^3/\mathbb{Z}_2$
   → use the matrix model representation of CS theory
Topological string side

Mirror curve to local $\mathbb{P}^1 \times \mathbb{P}^1$ (2D moduli space):

$$\Sigma : \{ H(x, y) = x^2 y + xy^2 + xy + q_1 y + q_2 x = 0 \}$$

Define coordinates [AKMV]

$$x_1 = 1 - \frac{q_1}{q_2}, \quad x_2 = \frac{1}{\sqrt{q_2(1 - \frac{q_1}{q_2})}}$$

Small radius point: $x_1, x_2 \to 0$. 
Generating the amplitudes

- **Disk** amplitude: just a function, so simply expand around $x_1, x_2 = 0$

- **Annulus** amplitude: Bergman kernel is quasi-modular, hence transforms with a shift under modular transformations
  - Large radius $\to$ small radius is an $S$-transformation (exchanging the $A$ and $B$ periods)
  - From quasi-modular properties we extract $B^{orb}(x_1, x_2)$ from large radius Bergman kernel

- Once we have disk and annulus, we generate everything else at the small radius point!
Chern-Simons theory on $S^3/\mathbb{Z}_2$

Given by a two-matrix model [Mariño,AKMV]:

$$Z(N_1, N_2, g_s) = \int dM_1 dM_2 \exp \left( -\frac{1}{2g_s}(\text{Tr} M_1^2 + \text{Tr} M_2^2) + V(M_1) + V(M_2) + W(M_1, M_2) \right)$$

with

$$V(M) = \frac{1}{2} \sum_{k=1}^{\infty} \frac{B_{2k}}{k(2k)!} \sum_{s=0}^{2k} (-1)^s \binom{2k}{s} \text{Tr} M^s \text{Tr} M^{2k-s}$$

$$W(M_1, M_2) = \sum_{k=1}^{\infty} \frac{B_{2k}(2^{2k} - 1)}{k(2k)!} \sum_{s=0}^{2k} (-1)^s \binom{2k}{s} \text{Tr} M_1^s \text{Tr} M_2^{2k-s}$$

(the matrix model can be written in a simpler form using Log potentials)
To compare with open topological string amplitude we need the expectation value of the unknot in Chern-Simons theory on $S^3/\mathbb{Z}_2$:

$$W_R(N_1, N_2, g_s) = \frac{1}{Z(N_1, N_2, g_s)} \langle \text{Tr} R e^M \rangle,$$

where, in terms of eigenvalues $m^1_i, m^2_j$ of $M_1, M_2$

$$e^M = \text{diag}(e^{m^1_1}, \ldots, e^{m^1_{N_1}}, -e^{m^2_1}, \ldots, -e^{m^2_{N_2}})$$

In this two-matrix model this can be computed perturbatively in the 't Hooft variables $S_i = g_s N_i$, $i = 1, 2$
Then, we can check that [BKMP]:

\[
\text{Perturbative expansion of the unknot in CS on } S^3/\mathbb{Z}_2 = \text{Open A-model amplitudes of local } \mathbb{P}^1 \times \mathbb{P}^1 \text{ at small radius}
\]

On the other hand:

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\text{Resummation at strong 't Hooft coupling of CS on } S^3/\mathbb{Z}_2 = \text{A-model amplitudes of local } \mathbb{P}^1 \times \mathbb{P}^1 \text{ at large radius}
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- The resummation involves a complicated moduli space (the complex structure moduli space of the mirror of local \( \mathbb{P}^1 \times \mathbb{P}^1 \), or the stringy Kähler moduli space of local \( \mathbb{P}^1 \times \mathbb{P}^1 \)), and a clever parameterization is need (choice of flat coordinates)
Then, we can check that [BKMP]:

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### Open questions

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Can we learn anything about AdS/CFT from this toy model?

So we have a complete B-model formalism for mirrors to toric geometries, which is nonperturbative in the moduli.

Many open questions:

- Proof of the recursion in the B-model? (beyond [DV])
- Matrix model duals for topological strings on all toric geometries?
- Compact threefolds?

Thank you! :-}
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highly curved string target ↔ perturbative gauge theory
large radius string theory ↔ strong coupling resummation

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Thank you! :-(