# Pseudo-defects, Fermion number and induced stability

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# 1 Outline

- Topological solutions with coupled fermions
- Fractional fermion number and stability theorem
- Toroidal geometry and stabilized vortons
    - Cosmological implications

# 2 Main results :

Usual situation :

i. Soliton a bosonic solution stabilised due to topology

ii. Fermions coupled to such may possess zero energy solutions localised on the soliton

iii. Zero energy solutions induce fermionic charge on the soliton

iv. The induced charge – fermion number – can be fractional, typically$\pm\frac{1}{2}, \pm\frac{3}{2}, \ldots$. shifting the spectrum by $\frac{1}{2}$

# 1 Related obervations

i. Zero energy solutions depend only on boundary conditions

ii. Topological stability of the background bosonic configuration not required

iii. What happens if the induced fermion number is fractional for a *meta-stable* bosonic configuration?

## 2 Things to investigate

i. Conditions for metastable solutions (meta-soliton)

ii. Existence of fermion zero modes such as to induce fractionl charge

iii. Check explicitly that in this case, tunnelling decay of metasoliton leads to physically unacceptable results

## 3 Further challenges

1. What happens if the fermionic species is Majorana?
   No distinct anti-particle so what do $-\frac{1}{2}, -\frac{3}{2}$ etc. mean?
   – Instead of a Dirac Sea one gets a small *Majorana Pond* of states with negative fermion number localised on the (meta-)soliton

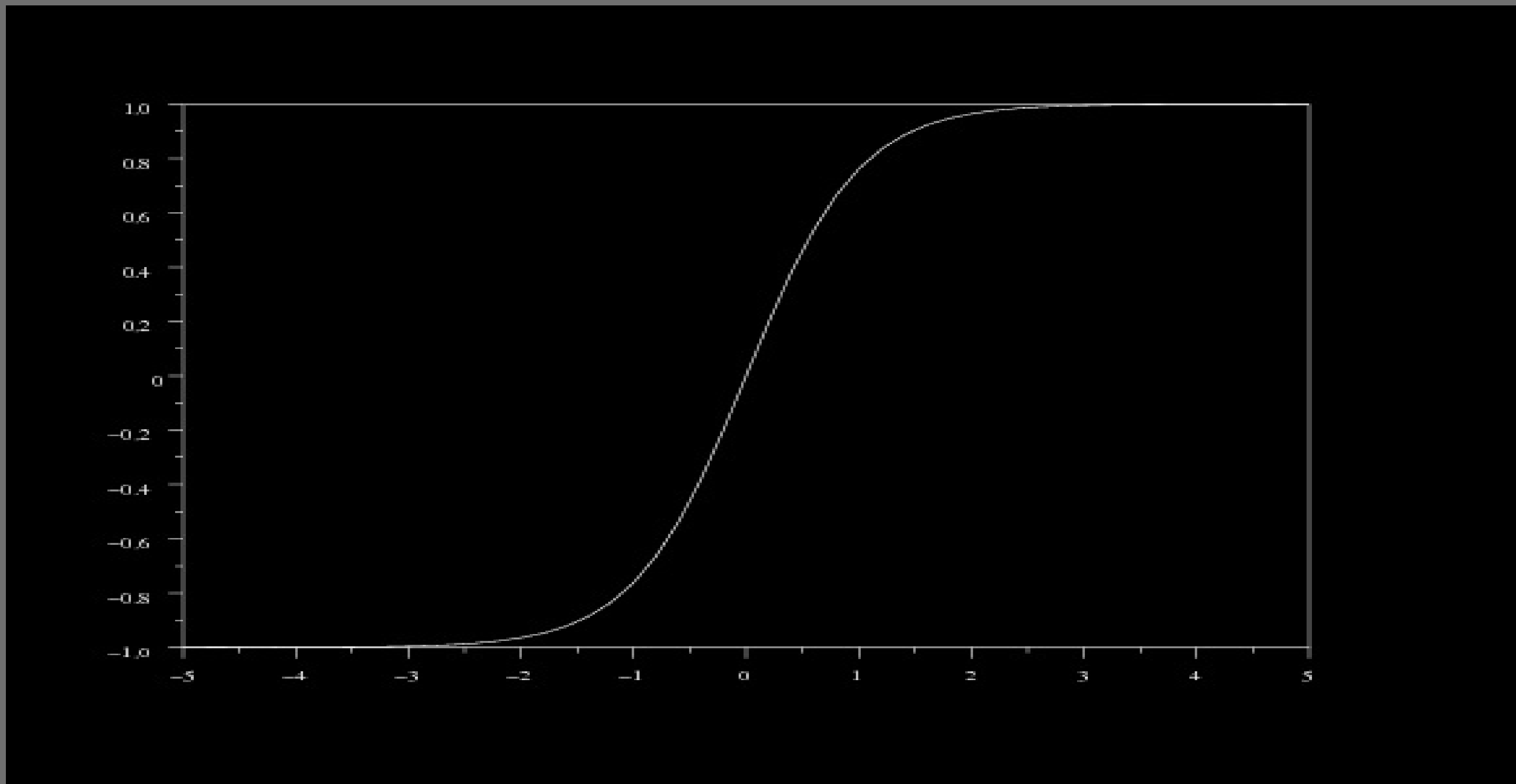
2. What happens if the metastability results from closed loop?
   – Hopf number steps in! Ensures stability

~~ End of summary of resuts ~~

## 2.1 An elementary example

The kink in $1+1$ dimensions

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}\lambda(\phi^2 - m^2)^2$$



Fermion field $\psi$ gets mass from the symmetry breaking

$$\mathcal{L}_\psi = i\bar{\psi}\gamma^\mu\partial_\mu\psi - h\phi\bar{\psi}\psi$$

Look for time-independent, ie "zero-energy" solution

$$\sigma^3\frac{\partial}{\partial x}\psi - h\phi_{\mathrm{bg}}(x)\psi = 0 \qquad h > 0$$

We find

$$\psi(x) \sim \begin{pmatrix} u_0 \exp\{h\int_{-\infty}^{x}\phi_{\mathrm{bg}}(x')\mathrm{dx}'\} \\ \\ v_0 \exp\{-h\int_{-\infty}^{x}\phi_{\mathrm{bg}}(x')\mathrm{dx}'\} \end{pmatrix}$$

So we can accept only one solution

$$\psi(x) = v_0 \begin{pmatrix} 0 \\ \\ \{\cosh(m\sqrt{\frac{\lambda}{2}}x)\}^{-h\sqrt{2/\lambda}} \end{pmatrix}$$

This is very unusual because usually every solution in field theory come paired and requires the interpretation as particle-antiparticle pair.
We shall now show that presence of zero modes induces fermion number on the soliton, which can be fractional.

Similarly it can be shown that fermionic zero modes exist on the Nielssen-Olesen string. We shall see this in a general setting later.

# 3 Fractional fermion number :

For a Dirac field we have the conserved number oprator satisfying

$$[N,\psi]=-\psi \qquad \text{and} \qquad [N,\psi^\dagger]=\psi^\dagger$$

Further, it obeys the charge conjugation property

$$\mathcal{C}\,N\mathcal{C}^\dagger=-N$$

In the presence of a zero-mode $\psi_0$ the field is expanded as

$$\psi=c\psi_0+\left\{\sum_{\kappa,s}a_{\kappa,s}\chi_{\kappa,s}(x)+\sum_{k,s}b_{k,s}u_{k,s}(x)+h.c.\right\}$$

Note that hermitian conjugates of the operators $a_{\kappa,s}$ and $b_{k,s}$ are included above in the "$h.c.$" but not that of the solitary mode operator $c$. The conjugate operator $c^\dagger$ appears separately in the expansion for $\bar{\psi}$.

Now the singleton zero mode is a real function and hence goes into itself under charge conjugation. However for the operator $c$ we must demand

$$\mathcal{C} c \mathcal{C}^\dagger = c^\dagger \qquad \text{and} \qquad \mathcal{C} c^\dagger \mathcal{C}^\dagger = c$$

But we would also like the vacuum to be charge conjugation invariant

$$\mathcal{C}\,|\text{vac}\rangle = \eta_{\text{vac}} |\text{vac}\rangle$$

upto a phase $\eta_{\text{vac}}$. This is problematic for interpreting one of $c$ or $c^\dagger$ as a creation operator and the other as destruction operator

$$c^\dagger |\text{vac}\rangle = |1\rangle \;\; \Rightarrow \;\; \mathcal{C} c^\dagger \mathcal{C}^\dagger \mathcal{C}\,|\text{vac}\rangle = |1\rangle$$

$$\Rightarrow \quad \eta_{\mathrm{vac}} c|\mathrm{vac}\rangle = |1\rangle$$

The only non-trivial irreducible realization of this algebra is to require the existence of a doubly degenerate ground state with states $|-\rangle$ and $|+\rangle$ satisfying

$$c|-\rangle = |+\rangle \qquad \text{and} \qquad c^{\dagger}|+\rangle = |-\rangle \tag{1}$$

with the simplest choice of phases. Now we find

$$\mathcal{C} c \mathcal{C}^{\dagger} \mathcal{C}|-\rangle = \mathcal{C}|+\rangle \tag{2}$$

$$\Rightarrow c^{\dagger}(\mathcal{C}|-\rangle) = (\mathcal{C}|+\rangle) \tag{3}$$

These requirements have the simplest non-trivial solution

$$\mathcal{C}|-\rangle = \eta_D^{-}|+\rangle \qquad \text{and} \qquad \mathcal{C}|+\rangle = \eta_D^{+}|-\rangle \tag{4}$$

### 3.0.1 Majorana case

The Majorana condition on a fermion field requires that we demand

$$\mathcal{C}c\mathcal{C}^\dagger = \pm c \qquad \text{and} \qquad \mathcal{C}c^\dagger\mathcal{C}^\dagger = \pm c^\dagger \tag{5}$$

(sign as in the Majorana condition; also works for more general choices). Unlike the Dirac case, the $c$ and $c^\dagger$ are not exchanged under charge conjugation. Again we must demand the existence of a doubly degenerate ground state with states $|-\rangle$ and $|+\rangle$ satisfying

$$c|-\rangle = |+\rangle \qquad \text{and} \qquad c^\dagger|+\rangle = |-\rangle \tag{6}$$

with the simplest choice of phases. Now we find

$$\mathcal{C}c\mathcal{C}^\dagger\mathcal{C}|-\rangle = \mathcal{C}|+\rangle \tag{7}$$

$$\Rightarrow \pm c(\mathcal{C}|-\rangle) = (\mathcal{C}|+\rangle) \tag{8}$$

This relation has the simplest non-trivial solution

$$\mathcal{C}|-\rangle = \eta_M^-|-\rangle \qquad \text{and} \qquad \mathcal{C}|+\rangle = \eta_M^+|+\rangle \tag{9}$$

In either case the number operator becomes fractional. Returning to the Dirac case, the standard fermion number operator

$$N_F = \frac{1}{2}(\psi_L^\dagger \psi_L - \psi_L \psi_L^\dagger) \tag{10}$$

acting on these two states gives,

$$\frac{1}{2}(c\,c^\dagger - c^\dagger c)|\pm\rangle = \pm\frac{1}{2}|\pm\rangle \tag{11}$$

The number operator indeed lifts the degeneracy of the two states. For $s$ number of zero modes, the ground state becomes $2^s$-fold degenerate, and the fermion number takes values in integer steps ranging from $-s/2$ to $+s/2$. For $s$ odd the values are therefore half-integral.

## 3.1 Majorana zero modes :

- If $n$ is odd then the induced Majoranan fermion number is half-integral($\pm n/2$).

- What is the meaning of -ve fermion number of the Majorana fermions ?

- Note, this conclusion is shown to exist in the zero-energy sector only.

- This leads to a conclusion that instead of Dirac sea continuing indefinitely into the negative values, we have a Majorana pond at the ground state.

- This degeneracy of the ground state can be thought of as spontaneous symmetry breaking for fermions.

# 4 Topological pseudo-defects

What are pseudo-defects?

Ila Garg and UAY arXiv:1802.03915

- Parent group is simply connected
- After symmetry breaking the vacuum manifold is disconnected
- The disjoint sets of vacuum manifold can be connected by a path in the original group
  - but at an energy cost in the low energy theory
- Analogous to two-stage symmetry breaking situations (eg. Peskin and Vilenkin 1993)
  - First stage of symmetry breaking gives rise to a discrete vacuum manifold
  - The second stage of breaking has domain walls
- but here there is only one step breaking and the parent group is simply connected

## 4.1 Pseudo-defects in SUSY SO(10)

SUSY SO(10) Model of

Aulakh, Bajc, Melfo, Senjanovic, Vissani 2004

The Higgs content 210 – $\Phi_{ijlk}$, 126($\overline{126}$) – $\Sigma_{ijklm}$ ($\bar{\Sigma}_{ijklm}$), (and additionally $H_i$)

The $\overline{126}$ with 210 breaks the SO(10) $\rightarrow$ MSSM .

The renormalizable superpotential

$$\begin{aligned} W &= \frac{m_\Phi}{4!}\Phi^2 + \frac{\lambda}{4!}\Phi^3 + \frac{m_\Sigma}{5!}\Sigma\,\bar{\Sigma} + \frac{\eta}{4!}\Phi\,\Sigma\,\bar{\Sigma} + m_H H^2 \\ & \quad + \frac{1}{4!}\Phi\, H\,(\gamma\,\Sigma + \bar{\gamma}\,\bar{\Sigma})\,. \end{aligned}$$

D-parity is defined as

$$D = \exp(i\,\pi\,J_{23})\exp(i\,\pi\,J_{67}) \tag{12}$$

This parity exchanges $\mathrm{SU}(2)_L \otimes U(1)_Y$ with $\mathrm{SU}(2)_R \otimes U(1)_{\tilde{Y}}$

But any continuous path joining the two necessarily passes through configurations of higher energy.

#### 4.1.1 Details of the breaking

In terms of SO(10) vector indices the fields which acquire vev are written as

$$
\begin{aligned}
P &= [7,8,9,10] \\
A &= [1,2,3,4]+[1,2,5,6]+[3,4,5,6] \\
W_R^0 &= [1,2,7,8]+[3,4,7,8]+[5,6,7,8]+[1,2,9,10]+[3,4,9,10]+[5,6, \\
&\quad 9,10] \\
\Sigma_R^- &= -i([1,3,5,7,9]-[2,4,5,7,9]-[2,3,6,7,9]-[1,4,6,7,9] \\
&\quad -i[2,3,5,7,9]-i[1,4,5,7,9]-i[1,3,6,7,9]+i[2,4,6,7,9]) \\
&\quad -(7,9\rightarrow 8,10)+i\,\{7,9\rightarrow 7,10\}+i\,\{7,9\rightarrow 8,9\} \\
\bar{\Sigma}_R^+ &= i([1,3,5,7,9]-[2,4,5,7,9]-[2,3,6,7,9]-[1,4,6,7,9] \\
&\quad +i[2,3,5,7,9]+i[1,4,5,7,9]+i[1,3,6,7,9]-i[2,4,6,7,9]) \\
&\quad -(7,9\rightarrow 8,10)-i\,\{7,9\rightarrow 7,10\}-i\,\{7,9\rightarrow 8,9\} \qquad (13)
\end{aligned}
$$

The sign $+(-)$ in the superscript represents the $T_{3R}$ value. Under the action of D-parity these vevs transform as

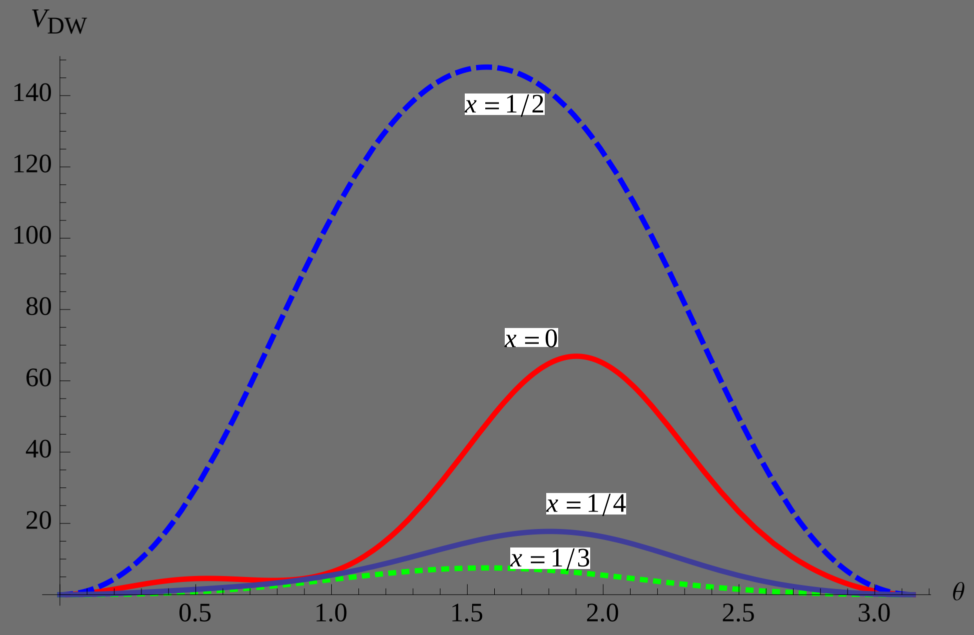
$$P \rightarrow -P; \quad A \rightarrow A; \quad W_R^0 \rightarrow W_L^0$$
$$\Sigma_R^- \rightarrow -\Sigma_L^+; \quad \bar{\Sigma}_R^+ \rightarrow -\bar{\Sigma}_L^-$$

### 4.1.2 Tracing out the barrier

- Both $SU(2)_L \otimes U(1)_Y$ and $SU(2)_R \otimes U(1)_{\tilde{Y}}$ can be seen to be local minima
- Both can be proved to be isolated. There are no zero energy displacements
- We propose a simple path joing the two

$$U(\theta)_D = \exp\{i\theta(J23 + J67)\}$$

Then we can trace (with cumbersome numerics) the barrier




The MSSM case corresponds to $x = 1/4$. Other $x$ values correspond to the barriers in other patterns of breaking. Each case is an isolated local SUSY preserving minimum with superpotential parameters tuned to give desired scheme of breaking.

# 5 Fermion zero modes on metastable strings

In either of the cases,

→ Metastable objects in sequential symmetry breaking

→ Topological pseudo-defects

Topological defects of low energy theory can decay by tunnelling (Preskill and Vilenkin; Kibble, Lazarides and Shafi).

But consider next, fermion zero modes on such objects

- In specific models Majorana fermions result from chiral fermioninc content in a way that in the vortex sector the interaction of fermions and the vertex is given by

$$\mathcal{L}_{\sigma - N_R} \sim h\eta_2 f(r)(e^{-in\phi}\overline{N_R^c}N_R + h.c.)$$

where $n$ is the winding number.

- The index of the zero modes is given by ($I \equiv n_L - n_R$). Here $n_L$ and $n_R$ presenting the number of left and right moving zero modes.
- The index can be computed from the formula

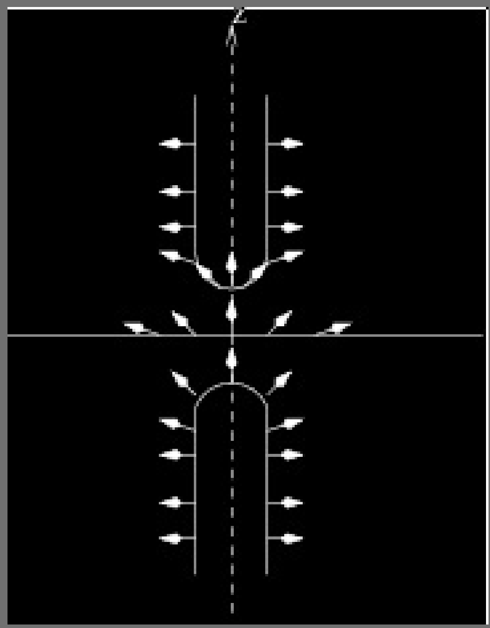
$$\mathcal{I} = \frac{1}{2\pi i}(\ln \det M)|_{\phi=0}^{2\pi}$$

(E. Weinberg; Ganoulis and Lazarides; Starkmann, Stojkovic and Vachaspati)

Thus zero modes are possible on metastable objects. If the number of zero modes is odd, the induced fermion number is half-integer, not corresponding to any free particle excitation in the bulk medium.

## 5.1 Quantum mechanical stability ( argument 1)

If the metastable string ruptures, each segment then can indefinitely shrink and disappear. But then the half-integer fermion number is not accounted for.



Thus our conclusion is that the rupture of the string into two segments (e.g. Preskill and Vilenkin) ending in monopoles has to be prevented by virtue of Quantum Mechanics. This has to be true for both Dirac and Majorana fermions.

## 5.2 Quantum mechanical stability ( argument 2)

Quantum Mechanical stability of this sector follows from distinctness of sectors of different values of $(-1)^{N_F}$.

Majorana fermions can be assigned a unique parity either of the values $\pm i$. Accordingly let us choose $i$ to be the parity of the free single fermion states in the trivial vacuum.

### 5.2.1 Parities of the zero-energy states

Parities of string localised states related to those of the free particle states

A free fermions is capable of being absorbed by the vortex (see for instance Davis, Perkins and Davis). In the zero energy sector this absorption would cause a transition from $|-\frac{1}{2}\rangle$ to $|+\frac{1}{2}\rangle$ and cause a change in parity by $i$. Thus the level carrying fermion number $+\frac{1}{2}$ should be assigned a parity $e^{i\pi/2}$ relative to the $-\frac{1}{2}$ state.

Symmetry between the two states suggests that we assign parity $e^{i\pi/4}$ to the $N_F = 1/2$ and $e^{-i\pi/4}$ to the $N_F = -1/2$ states.

### 5.2.2 Superselection rule

We now show the inappropriateness of superposing states of half-integer valued fermion number and integer valued fermion number (Wick, Wightman and Wigner; S. Weinberg). The operation $\mathcal{P}^4$, parity transformation performed four times must return the system to the original state, upto a phase.

Consider forming the state $\Psi_S = \frac{1}{\sqrt{2}}(|\frac{1}{2}\rangle + |1\rangle)$ from states of half-integer and integer value for the fermion number. But

$$\mathcal{P}^4 \Psi_S = \frac{1}{2}(-|\frac{1}{2}\rangle + |1\rangle)$$

Thus this operation identifies a state with another orthogonal to it.

Similarly, application of $\mathcal{P}^2$ which should also leave the physical content of a state unchanged.

$$\mathcal{P}^2\Psi_S = \frac{1}{\sqrt{2}}(i|\frac{1}{2}\rangle - |1\rangle)$$

There can be no meaningful operator possessing non-trivial matrix elements between the two spaces.

# 6 Toroidal cosmic string :

This is a configuration which is expected to result naturally for the same Lagrangian as Nielssen and Olesen; Jackiw and Rossi. It should become the famous $2+1$ dimensional solution for long length.

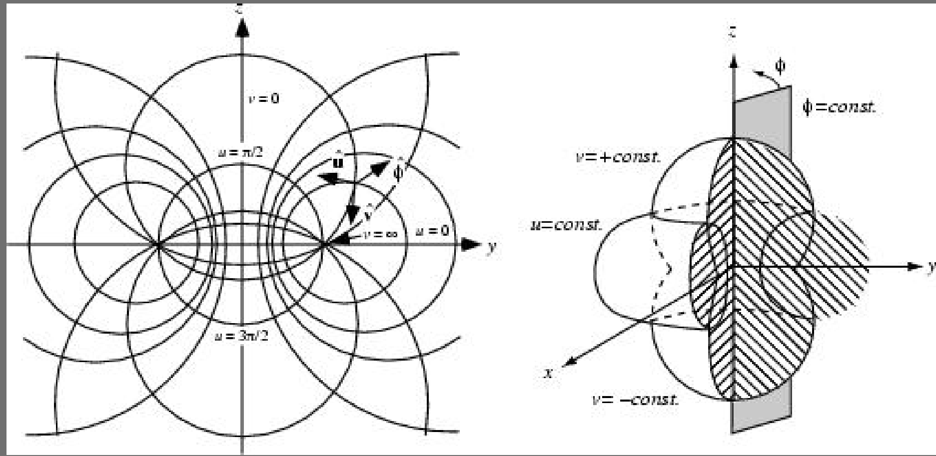
To determine zero modes in this case, we instroduce toroidal coordinates $(v, u, \varphi)$ given by

$$x = \frac{a \sinh v \cos\varphi}{\cosh v - \cos u} \tag{14}$$

$$y = \frac{a \sinh v \sin\varphi}{\cosh v - \cos u} \tag{15}$$

$$z = \frac{a \sin u}{\cosh v - \cos u} \tag{16}$$

where $v$ ranges from 0 to $\infty$, $u$ ranges from 0 to $2\pi$ and $\varphi$ ranges from 0 to $2\pi$.




Then the Dirac (Majorana) equation becomes

$$\begin{bmatrix} -e^{i\varphi}[D_r + \frac{i}{r}D_\varphi] & D_z + D_t \\ D_z - D_t & e^{-i\varphi}[D_r - \frac{i}{r}D_\varphi] \end{bmatrix} \begin{bmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{bmatrix} = g_Y\phi \begin{bmatrix} \tilde{\psi}_1^* \\ \tilde{\psi}_2^* \end{bmatrix} \tag{17}$$

However $\varphi$ explicitly appears in the equations for $\tilde{\psi}$ and factoring out this dependence requires us to introduce the ansatz $\tilde{\psi}_1 = e^{-i\varphi/2}\psi_1$ and $\tilde{\psi}_2 = e^{i\varphi/2}\psi_2$. This amounts to anti-periodic boundary condition appropriate to a fermion as we traverse the length of the loop. Then the equations obeyed by $\psi_1$ and $\psi_2$ are

$$\begin{bmatrix} -[D_r + \frac{1}{2r}] & D_z \\ D_z & [D_r + \frac{1}{2r}] \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = g_Y \phi \begin{bmatrix} \psi_1^* \\ \psi_2^* \end{bmatrix} \tag{18}$$

$$[\sin\xi \sin\bar{\xi}(\frac{\partial}{\partial\xi} + i(\frac{n}{2}g - (n-l))) - i\frac{\sin^2\bar{\xi}}{2\sinh v}]Y = \frac{1}{2}a\,k\,m\,X \tag{19}$$

### 6.1 Counting the zero modes :

To count the total number of fermion zero modes present on soliton in $n$ vortex sector, we observe the $v$ dependance of the solution near the loop i.e. as $v \to \infty$. As mentioned above $g$ and $k$ both tend to zero as we approach the loop. So substituting $g = 0 = k$ the equations (?) and (19), near the loop, respectively become,

$$(\frac{d}{dv} + l)X = 0 \tag{20}$$

$$(\frac{d}{dv} + (n - l))Y = 0 \tag{21}$$

So, $X \sim e^{-lv}$and $Y \sim e^{-(n-l)v}$. And so the behaviour of $\psi$ near the loop is,

$$\psi = \sin\xi(X e^{ilu} + Y^* e^{i(n-l)u}) = \sin\xi(C_1 e^{2il\xi} + C_2 e^{2i(n-l)\xi}) \tag{22}$$

In the limit $v \to \infty$ the right hand side of eq. (22) is dominated by the terms

$$\psi \longrightarrow \frac{1}{2i}(C_1 e^{i2(l-\frac{1}{2})\xi} + C_2 e^{i2[(n-1)-(l-\frac{1}{2})]\xi}) \tag{23}$$

Recalling $\xi = (u + i\,v)\,/\,2$ and denoting $l - \frac{1}{2}$ by $l'$, and requiring $\psi$ to remain finite near the loop i.e. as $v \to \infty$, we need,

$$0 \leq l' \leq (n-1) \tag{24}$$

This gives us total of $n$ complex normalisable solutions, the same result as for the infinitely long string.

## 6.2 Induced stability of string loop

We are again led to the conclusion that there can be cloops with odd number of zero modes which cannot shrink indefinitely and disintegrate – in doing so it would have to violate fermion number by a fractional value.

# 7 Implicaions to cosmology and unification :

Strings endowed with fractional fermion number will not scale as learnt from simulations of bosonic fields alone.

Quantum mechanically stabilised loops may

- be over-abundant and conflict with cosmology
- have just enough abundance to explain Dark Matter

# 8 Conclusions

- There exist topological pseudo-defects (TPDs) due the nature of Higgs representation
- In SO(10) case TPDs could have signatures on the inflation observables.
- Not all meta-stable topological objects may actually be so

- Fermions, even treated as quantum perturbation can stabilize such semi-classical objects
- The result persists even if the fermion number is not a conserved quantity

"Stability without stability"
or
"Conservation without a conservation law"
(courtesy J. A. Wheeler)

Thank you

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