Helicity and the Dissipation Of Energy in Incompressible Fluids

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Lightning Review of Fluid Mechanics

- When the fluid speed is small compared to the speed of sound, density may be treated as a constant.
- Then conservation of mass reduces to $\mathrm{div}\ v = 0$
- Equating the acceleration of a small cube of fluid to the force divided by mass gives

$$\frac{\partial v}{\partial t} + v \cdot \nabla v = -\nabla p + \nu \nabla^2 v, \quad \mathrm{Navier-Stokes\ Equations}$$

- Here, p is the pressure divided by density and ν the kinematical viscosity (which has units of $\frac{\mathrm{length}^2}{\mathrm{time}}$).

Vorticity

- Eliminate pressure by taking a curl.
- Vorticity $\omega = \mathrm{curl}\ v$
- Use the identity $\nabla\left(\frac{1}{2}v^2\right) = v\cdot\nabla v + v\times\omega,$
- $\frac{\partial\omega}{\partial t} + \mathrm{curl}\ [\omega\times v] = \nu\ \nabla^2\omega$ The Vorticity form of NS equations

Energy

- Define the velocity potential $\psi$by $v = \mathrm{curl}\ \psi$
- $\psi \cdot \omega = v^2 + \text{total divergence}$
- $\nabla^2 \omega = -\mathrm{curl}^2 \omega, \quad \psi \cdot \nabla^2 \omega = -\omega^2 + \text{total divergence}$
- Multiplying the vorticity for of NS equations by ψ and integrating
- $\frac{d}{dt} \int \frac{1}{2} v^2 d^3x = -\nu \int \omega^2 d^3x$ Dissipation of energy
- LHS is the rate of change of kinetic energy divided by density.
- An incompressible fluid has no potential energy

Lie Algebra of Vector Fields

- A vector field should be thought of as a first order differential operator on scalar fields: $v(f) = v \cdot \nabla f$
- A natural operation is the commutator of such operators: $[u, v] = u \cdot \nabla v - v \cdot \nabla u$
- This is anti-symmetric and satisfies the Jacobi identity $[[u, v], w] + [[v, w], u] + [[w, u], u] = 0.$
- The set of vector fields is a Lie algebra
- An Identity : $\mathrm{curl}(u \times w) = [w, u] + u \, \mathrm{div} w - w \, \mathrm{div} u$
- The commutator of two incompressible vector fields is also incompressible

The Lie Algebra $\mathfrak{s}(3)$

- The Lie algebra $\mathfrak{s}(3)$ of incompressible vector fields is the phase space of fluid mechanics $[u, w] = \mathrm{curl}\,(w \times u)$
- Commutator form of NS equations $\frac{\partial\omega}{\partial t} + [v, \omega] = \nu\ \nabla^2\omega$
- Euler's equation is the limit of zero viscosity $\frac{\partial\omega}{\partial t} + [v, \omega] = 0$
- This ideal fluid equation is remarkably similar to another equation of Euler: the rigid body

The Rigid Body

- Euler Equation for a rigid body $\frac{dL}{dt} + \Omega \times L = 0, \quad L = G\Omega$
- L is angular momentum, Ω is angular velocity and G is the moment of inertia
- The cross product of vectors is anti-symmetric and satisfies the Jacobi identity $(a \times b) \times c + (b \times c) \times a + (c \times a) \times b = 0$.
- Thus it turns the three-dimensional space of vectors into a Lie algebra.
- This is just $\mathfrak{so}(3)$, the Lie algebra of infinitesimal rotations.

The Invariant Inner Product in $\mathfrak{so}(3)$

- The scalar product of vectors (“dot product”) satisfies the condition of being an invariant inner product of the Lie algebra $\mathfrak{so}(3)$:
- $a \cdot (b \times c) + (b \times a) \cdot c = 0$
- Setting $a = \Omega, b = L, c = L$ gives us $L \cdot (\Omega \times L) = 0$.
- This leads to a conserved quantity, the quadratic Casimir implied by the invariant inner product $\frac{d}{dt}\left(\frac{1}{2} L \cdot L\right) = 0$
- In addition, energy is conserved $\frac{dE}{dt} = 0, \quad E = \frac{1}{2}\Omega \cdot L$
- The rigid body is a hamiltonian system with the energy as the hamiltonian. $\{f, g\} = L \cdot (\nabla_L f \times \nabla_L g)$
- $\{E, f\} = L \cdot (\Omega \times \nabla_L f) = -(\Omega \times L) \cdot \nabla_L f = \frac{df}{dt}$

Helicity is The Invariant Inner Product in $\mathfrak{s}(3)$

- Define an inner product in $\mathfrak{s}(3)$(See Arnold Khesin *Topological Methods in Hydrodynamics*) $\langle u, w\rangle = \int u \cdot \mathrm{curl}^{-1} w \; dx$
- Using $\mathrm{curl}\; w \times u = [u, w]$, and the anti-symmetry of the triple product of vectors,

$$\langle u, [w, s]\rangle = \int u \cdot (s \times w)\, dx = -\langle [w, u]\, , s\rangle.$$

- Thus $\langle , \rangle$ is an invariant inner product in $\mathfrak{s}(3)$.
- So there is a quadratic Casimir that is conserved in ideal incompressible fluid flow: $\mathcal{H} = \frac{1}{2} \int v \cdot \omega dx = \frac{1}{2}\langle \omega, \omega\rangle$
- $\frac{d}{dt}\mathcal{H} = 0$
- This quantity is called Helicity. It measures (H. K. Moffat) the average linking number of vortex lines (i.e., integral curves of vorticity).

Summary of the Analogy of Rigid Body to Ideal Fluid

	Rigid Body	Ideal Incompressible Fluid
velocity	Ω	v
(angular) momentum	L	ω
inertia	G	curl
Phase Space	$\mathfrak{so}(3) \equiv \mathbb{R}^3$	$\mathfrak{s}(3) = \{\mathbf{v} \mid \mathrm{div}\ \mathbf{v}=0\}$
Lie product	$a \times b$	$[A, B] = A \cdot \nabla B - B \cdot \nabla A$
Invariant inner product	$a \cdot b$	$\langle u, w \rangle = \int u \cdot \mathrm{curl}^{-1} w dx$
Equation of motion	$\frac{dL}{dt} + \Omega \times L = 0$	$\frac{\partial \omega}{\partial t} + [v, \omega] = 0$
Energy	$E = \frac{1}{2} \Omega \cdot L$	$E = \frac{1}{2} \int v \cdot v dx = \frac{1}{2} \langle v, \omega \rangle$
Casimir	$C = \frac{1}{2} L \cdot L$	$\mathcal{H} = \frac{1}{2} \int v \cdot \omega dx = \frac{1}{2} \langle \omega, \omega \rangle$
Poisson Bracket	$\{f, g\} = L \cdot (\nabla_L f \times \nabla_L g)$	$\{f, g\} = \langle \omega, [\nabla_\omega f, \nabla_\omega g] \rangle = \int \omega \cdot \left(\frac{\delta f}{\delta v} \times \frac{\delta g}{\delta v} \right) dx$

Abelian Gauge Fields are Dual to Incompressible Flows

- Given a vector field v and a 1-form A we can form a scalar field by contraction $i_v A$.
- Given also a density (equal to one in our choice of co-ordinates) we can integrate this scalar field to get a number: $\int i_v(A)dx$.
- In terms of components this is just $\int v^i A_i dx$.
- If we were to change the 1-form by an abelian gauge transformation $A \to A + d\Lambda$
- $\int v^i A_i dx \mapsto \int v^i A_i dx + \int v^i \partial_i \Lambda dx$
- Since $\partial_i v^i \equiv \mathrm{div}\ v = 0$, the second term is zero by integration by parts.
- The dual of the space of incompressible vector fields is the space of 1-forms modulo gauge transformations.

Helicity is the Chern-Simons Invariant

- The Chern-Simons invariant $\int A \wedge dA$ is clearly invariant under the action of the group of co-ordinate transformations (vector fields are infinitesimal co-ordinate transformations).
- It is also invariant under gauge transformations.
- In our previous notation, it is just $\int A \cdot \mathrm{curl}\; A\; dx$
- This is the contra-variant version (inverse of the covariant version) of the inner product on the Lie algebra:
- The Chern-Simons integral, is an invariant inner product in the dual of the Lie algebra.
- Setting $A = v$ we get Helicity.

Dissipation and Helicity

- Just as friction can dissipate angular momentum in a rigid body, helicity is not conserved with viscosity
- $\frac{d\mathcal{H}}{dt} = -\nu \int \omega \cdot \mathrm{curl}\ \omega dx$
- An important difference is that helicity is not positive, nor is its time derivative.
- Unlike energy, it is not necessary that its magnitude should decrease with time due to viscosity.
- Time evolution can make vortex lines more entangled, despite dissipation of energy.
- We can put a bound on the magnitude of helicity using the Cauchy-Schwarz inequality $\mid \frac{1}{2} \int v \cdot \omega dx \mid \leq \sqrt{\frac{1}{2} \int v^2 dx} \sqrt{\frac{1}{2} \int \omega^2 dx}$

A Dissipation Inequality-1

- Start with the inequality $\frac{1}{2}\left[v \pm \lambda\omega\right]^2 \geq 0$ where λ is some quantity with the dimensions of length.
- That is, $\frac{1}{2}v^2 \pm \lambda v \cdot \omega + \lambda^2 \frac{1}{2}\omega^2 \geq 0$.
- Integrate and use the dissipation equation for energy: $E \pm 2\lambda\mathcal{H} - \frac{\lambda^2}{2\nu}\frac{dE}{dt} \geq 0$.
- Multiply by $\frac{2\nu}{\lambda^2 E}$ and rearrange to get the differential inequality: $\frac{d\log E}{dt} \leq \frac{2\nu}{\lambda^2} - \frac{4\nu}{\lambda}\frac{|\mathcal{H}|}{E}$
- We can integrate this to get :$\log\frac{E(t)}{E_0} \leq \frac{2\nu}{\lambda^2}t - \frac{4\nu}{\lambda}\int_0^t \frac{|\mathcal{H}(t')|}{E(t')}dt'$
- Exponentiating, $E(t) \leq E_0 e^{\frac{2\nu}{\lambda^2}t} e^{-\frac{4\nu}{\lambda}\int_0^t \frac{|\mathcal{H}(t')|}{E(t')}dt}$

A Dissipation Inequality-2

- In the absence of viscosity, this is not a useful inequality: it says only that energy grows slower than an exponential.
- We already know that in this case it is a constant for zero viscosity!
- For $\nu > 0$,the inequality says that energy must decrease faster when the magnitude of helicity grows in comparison to energy.
- Helicity could grow in magnitude even with viscosity: the flow can become more intertwined with time.
- So $\frac{|\mathcal{H}(t)|}{E(t)}$ can become large: energy decreases while the magnitude of helicity can increase.
- Perhaps this is useful in controlling the growth of potential singularities in the solutions of NS equations.

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