Macroscopic tunnelling of two quantum spins with easy-axis anisotropy and an exchange interaction and the Haldane like spin chain

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The ground state of the Hamiltonian of two interacting spins

\[ H = -K (S^2_{1z} + S^2_{2z}) \pm \lambda \vec{S}_1 \cdot \vec{S}_2 \]

is not completely understood. For the plus sign we have an anti-ferromagnetic coupling, and for the minus sign it is ferromagnetic.
• We consider the case $\lambda/K \rightarrow 0$. For the decoupled theory the ground state is 4 fold degenerate:

$$|s, s\rangle, |s, -s\rangle, |s, -s\rangle, |-s, s\rangle$$

• We will write these as:

$$|\uparrow, \uparrow\rangle, |\downarrow, \downarrow\rangle, |\uparrow, \downarrow\rangle, |\downarrow, \uparrow\rangle$$

• with energy: $E = -2Ks^2$

• while the first excited states are 8 fold degenerate with energy splitting

$$\Delta E = K(2s - 1)$$
- In the weak coupling limit, it is interesting to ask what is the ground state and the first few excited states for large spin $\vec{S}$.

- For spin 1/2, the Hamiltonian is easily diagonalizable but it is a 4 x 4 matrix. For spin 1 the problem is also doable, but now it is a 9 x 9 matrix. In general one has a $(2s + 1)^2 \times (2s + 1)^2$ matrix to diagonalize. This quickly gets out of hand.

- One should do degenerate perturbation theory in the 4 dimensional subspace. But it is easy to see that the perturbation is already diagonal in the subspace.
• The two states $|\uparrow, \uparrow\rangle, |\downarrow, \downarrow\rangle$ are actually exact eigenstates of the full Hamiltonian. We can rewrite:

$$H = -K(S_{1z}^2 + S_{2z}^2) \pm \lambda \left( S_{1z}S_{2z} + \frac{1}{2}(S_{1}^+S_{2}^- + S_{1}^-S_{2}^+) \right)$$

• Then it is clear the raising and lowering operators annihilate these states. The energy is exactly $(-2K \pm \lambda)s^2$.

• The two states $|\uparrow, \downarrow\rangle, |\downarrow, \uparrow\rangle$ are not exact eigenstates, but the perturbation is diagonal in the subspace giving the energy

$$(-2K \mp \lambda)s^2$$
Thus for the lower sign, the ferromagnetic case, the exact ground state is doubly degenerate and the first excited states are the up-down and down-up combinations.

For the upper sign, the anti-ferromagnetic case, the up-down and down-up states seem to be the degenerate ground states, in lowest order degenerate perturbation theory, and the first excited states are the up-up and down-down states, which remain exact eigenstates.

Our analysis shows that in fact the up-down down-up degeneracy is broken by tunnelling, in both cases.

The symmetric combination \( \frac{1}{\sqrt{2}} (|\uparrow, \downarrow\rangle + |\downarrow, \uparrow\rangle) \) is split from the antisymmetric combination \( \frac{1}{\sqrt{2}} (|\uparrow, \downarrow\rangle - |\downarrow, \uparrow\rangle) \) due to tunnelling.
To obtain these results we use the spin coherent states path integral.

To get the states $|\pm\rangle = \frac{1}{\sqrt{2}}(|\uparrow, \downarrow\rangle \pm |\downarrow, \uparrow\rangle)$ through perturbation theory seems difficult.

Perturbation links the state $|\pm s, \mp s\rangle$ only to the state $|\pm s \mp 1, \mp s \pm 1\rangle$

To reach the state $|\mp s, \pm s\rangle$ requires one to go to $2s^{th}$ order in perturbation theory.

Then one would expect any admixture to be proportional to $\lambda^{2s}$.

Indeed we find the energy splitting is of this order, but the admixtures are order 1.
The spin coherent states path integral

- We use the path integral to compute the matrix element:
  \[ \langle \pm | e^{-\beta H} | \pm \rangle = \int \mathcal{D}\{\theta_i, \phi_i\} \, e^{-S_E} \]
  where the plus and minus signs are uncorrelated and the integration is done with appropriate boundary conditions.

- The Euclidean action is given by:
  \[ \mathcal{L}_E = i\dot{\phi}_1 (1 - \cos \theta_1) + V(\theta_1) + i\dot{\phi}_2 (1 - \cos \theta_2) + V(\theta_2) \]
  \[ + \lambda (\sin \theta_1 \sin \theta_2 \cos (\phi_1 - \phi_2) + \cos \theta_1 \cos \theta_2) \]
• The Feynman path integral in Minkowski space is not a well defined mathematical expression.
• The integral is not absolutely convergent.
• Consider the two dimensional example:

\[\int dx dy e^{i(x^2 + y^2)}\]

Changing variables to polar coordinates we have

\[2\pi \int_{0}^{\infty} dr r e^{ir^2} = (\pi/i)e^{ir^2}\bigg|_{0}^{\infty} = \infty\]
• The actual definition of the path integral is via the Euclidean path integral, with imaginary time.

\[ i S_{Mink.} \rightarrow -S_E \]
\[ t \rightarrow -i\tau \]
\[ \partial_t \rightarrow i\partial_\tau \]
\[ \partial_t \phi \partial_t \phi \rightarrow -\partial_\tau \phi \partial_\tau \phi \]
Then the Euclidean functional integral defined by:

\[ Z_E[J] = \frac{1}{\mathcal{N}} \int \mathcal{D}\phi \ e^{-S_E[\phi] + \int J \phi} \]
Complex Actions

• The Euclidean space action is sometimes not real.
• It can have parts which are imaginary.
• If the Minkowski action has a term which is t-odd, its analytic continuation to Euclidean space generally yields an imaginary term.
• Fermions contribute to the path integral with a factor that is real, but can be negative. This corresponds to an action which has an imaginary part.
• Complex actions come in many forms, but they usually contain topological terms.
• Chern-Simons terms
• Wess-Zumino terms
• epsilon tensor related expressions, for example the theta term in four dimensions:

\[ WZ = \frac{N}{24\pi^2} \int_{\frac{1}{2} S^5 + S^4} d^5 x \epsilon^{\mu\nu\lambda\sigma\tau} tr \left[ U^\dagger \partial_\mu U U^\dagger \partial_\nu U \cdots U^\dagger \partial_\tau U \right] \]

\[ CS = \lambda \int_{\mathbb{R}^3 + \infty} d^3 x \epsilon^{\mu\nu\lambda} tr \left[ A_\mu \partial_\nu A_\lambda + \frac{2}{3} A_\mu A_\nu A_\lambda \right] \]

\[ \sim \epsilon_{\mu\nu\lambda\rho} F^{\mu\nu} F^{\lambda\rho} \]
• such terms are linear in the time derivative
• hence the $i$ in front of the Minkowski space action is not cancelled, indeed:

$$\int dt \partial_t \rightarrow \int d\tau \partial_\tau$$

thus the Euclidean action is in general complex and the functional integral is of the form:

$$Z_E = \frac{1}{\mathcal{N}} \int \mathcal{D}\phi \ e^{-S_E[\phi] + iS_{top.}[\phi]}$$
• This is not an great problem to the proper mathematical definition of the functional integral.

• However, the usual perturbative paradigm of quantum mechanics, to find the classical critical points of the action and quantize the small oscillations, is not straightforward.

• Imagine that we have written the action strictly in terms of real fields, which is always possible.

• There are, in general, no solutions to the equations of motion.
• Classical solutions are the critical points of the action.
• The corresponding equations of motion have no solution for real fields in general
• Solutions may exist, but they are off the real axis in complexified field space.

\[
\frac{\delta S_E}{\delta \phi} + i \frac{\delta S_{top.}}{\delta \phi} = 0
\]
A ferromagnetic particle with a large magnetization is described by the Lagrangian of a particle on a sphere, with the addition of the Wess-Zumino term.

\[ S_{\text{Mink.}} = \int dt \left( \frac{I}{2} \partial_t \hat{s} \cdot \partial_t \hat{s} - V(\hat{s}) \right) + \sigma \int d^2 x \epsilon^{ij} (\hat{s} \cdot \partial_i \hat{s} \times \partial_j \hat{s}) \]

Which yields a Euclidean Lagrangian

\[ S_{\mu} = \int d\tau \left( \frac{I}{2} \partial_\tau \hat{s} \cdot \partial_\tau \hat{s} + V(\hat{s}) \right) - i \sigma \int d^2 x \epsilon^{ij} (\hat{s} \cdot \partial_i \hat{s} \times \partial_j \hat{s}) \]

Tunnelling is mediated by instantons, the solutions of the Euclidean equations of motion.
The second term is integrated over a two dimensional manifold whose boundary corresponds to the time variable.

For convenience we take this to be periodic, i.e. a circle. Then the 2-d manifold can be taken as half a 2 sphere, the equator of which is the time variable.
• For this case the Wess-Zumino term can be actually written as a local 1-d density
• We must fix a filling in for the half sphere
• We take:

\[ \hat{s} \equiv (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta) \]

\[ \theta(t, x) = (1 - x) \times \theta(t), \quad \varphi(t, x) = \varphi(t) \]

• And

\[ \partial_x \hat{s} = -\dot{\theta} \theta(t), \quad \partial_t \hat{s} = (1 - x) \dot{\theta} \dot{\theta} + \sin((1 - x)\theta) \dot{\varphi} \dot{\varphi} \]

\[ \hat{s} \cdot (\partial_t \hat{s} \times \partial_x \hat{s}) = \theta \sin((1 - x)\theta) \dot{\varphi} \]

• Then the integral over \( x \) can be done explicitly, giving:

\[ S_{WZW} = \sigma \int dt (1 - \cos \theta) \dot{\varphi} \]
• The potential is assumed to be easy-axis, reflection symmetric, with two classically degenerate minima at the two poles:

\[ V(\theta, \phi) \equiv V(\theta) = V(\pi - \theta) \]

\[ V(\hat{s}) \equiv V(\theta, \phi) = \frac{1}{2} \gamma \sin^2 \theta \]
Example of a suitable potential:
• Returning to our original problem, the Euclidean Lagrangian with: $\sigma_1 = \sigma_2 = s$

$$\mathcal{L}_E = is\dot{\phi}_1 (1 - \cos \theta_1) + V(\theta_1) + is\dot{\phi}_2 (1 - \cos \theta_2) + V(\theta_2)$$

$$+ \lambda (\sin \theta_1 \sin \theta_2 \cos (\phi_1 - \phi_2) + \cos \theta_1 \cos \theta_2)$$

• The equations of motion are

$$is \frac{d}{d\tau} (1 - \cos \theta_1) + \lambda \sin \theta_1 \sin \theta_2 \sin (\phi_1 - \phi_2) = 0$$

$$is \frac{d}{d\tau} (1 - \cos \theta_2) - \lambda \sin \theta_1 \sin \theta_2 \sin (\phi_1 - \phi_2) = 0$$

• and

$$\frac{\partial \mathcal{L}}{\partial \theta_1} = 0 = \frac{\partial \mathcal{L}}{\partial \theta_2}$$
• Adding the first two together gives:

\[
\frac{d}{d\tau} (\cos \theta_1 + \cos \theta_2) = 0.
\]

• We are interested with the initial condition \( \theta_1 = 0, \theta_2 = \pi \)

\[
\cos \theta_1 + \cos \theta_2 = l = 0
\]

• thus

\[
\Rightarrow \theta_2 = \pi - \theta_1
\]

\[
\theta = \theta_1, \phi = \phi_1 - \phi_2, \Phi = \phi_1 + \phi_2
\]

• Writing

• we get the effective Lagrangian

\[
\mathcal{L} = is\Phi - is\dot{\phi} \cos \theta + U(\theta, \phi)
\]

\[
U(\theta, \phi) = 2V(\theta) + \lambda (\sin^2 \theta \cos \phi - \cos^2 \theta) + \lambda
\]
Then the equations of motion become:

\[ i s \dot{\phi} \sin \theta = -\frac{\partial U (\theta, \phi)}{\partial \theta} \]

\[ i s \dot{\theta} \sin \theta = \frac{\partial U (\theta, \phi)}{\partial \phi} \]

These have no non-trivial solutions on the space of real functions. Multiplying the first by \( \dot{\theta} \) and the second by \( \dot{\phi} \) and subtracting yields

\[ \frac{dU (\theta, \phi)}{d\tau} = 0 \quad \text{i.e.,} \quad U (\theta, \phi) = \text{const.} = 0 \]

Taking the simple case \( V(\theta) = \gamma \sin^2 \theta \)

gives \( U (\theta, \phi) = (2\gamma + \lambda (\cos \phi + 1)) \sin^2 \theta \)
• Hence
\[ \cos \phi = \frac{-2\gamma}{\lambda} - 1 \]

• This has no real solution for \( \phi \)
\[ \phi = \phi_R + i\phi_I \]
\[ \cos \phi = \cos \phi_R \cosh \phi_I - i \sin \phi_R \sinh \phi_I \]
\[ \phi_R = n\pi \]

• The general solution is
\[ \cos \phi = (-1)^n \cosh \phi_I = \begin{cases} - \left( \frac{2K}{\lambda} + 1 \right) & \text{if } \lambda > 0 \\ + \left( \frac{2K}{|\lambda|} - 1 \right) & \text{if } \lambda < 0 \end{cases} \]

• \( n \) is 1 or 0 depending on the sign of \( \lambda \)
• This gives the unified expression

\[ \cosh \phi_I = \frac{2K + \lambda}{|\lambda|} \]

• as

\[ n = 1 \text{ for } \lambda > 0 \text{ and } n = 0 \text{ for } \lambda < 0 \]

• and

\[ \lambda(-1)^n = -|\lambda| \]

• the equation for \( \theta \) becomes:

\[ is \frac{\dot{\theta}}{\sin \theta} = -\lambda \sin \phi = -i\lambda(-1)^n \sinh \phi_I = i|\lambda| \sinh \phi_I \]
• This equation is easily integrated, we find

\[ \theta (\tau) = 2 \arctan \left( e^{\omega (\tau - \tau_0)} \right) \]

\[ \omega = (|\lambda|/s) \sinh \phi_I \]

• This is actually completely unimportant!
• The action for this instanton is exactly zero.
• The kinetic term and the potential vanish.

\[ S_0 = \int_{-\infty}^{\infty} d\tau \mathcal{L} = 0 \]
• But we still get a non-zero change in the action, since

\[
\Delta S = \int_0^{n\pi + i\phi_I} -is d\phi \cos \theta|_{\theta=0} + S_0 + \int_{n\pi + i\phi_I}^{0} -is d\phi \cos \theta|_{\theta=\pi} = -is 2n\pi + 2s\phi_I
\]

• The matrix element is given by the path integral

\[
\langle \theta_f, \phi_f | e^{-\beta H} | \theta_i, \phi_i \rangle = N \int_{\theta_i, \phi_i}^{\theta_f, \phi_f} D\theta D\phi \ e^{-S_E}
\]

• thus

\[
\langle \downarrow, \uparrow | e^{-\beta H} | \uparrow, \downarrow \rangle = N e^{-\Delta S} \kappa \beta (1 + \cdots)
\]

\[
N = Ne^{-E_0 \beta}
\]
• The contributions exponentiate

\[ e^{-\Delta S} \kappa \beta \rightarrow \sinh (e^{-\Delta S} \kappa \beta) \]

• Since \( \Delta S = -is2n\pi + 2s\phi_I \)

• and

\( \phi_I \) for \( K \gg |\lambda| \)

\[ \phi_I = \arccosh \left( \frac{2K + \lambda}{|\lambda|} \right) \approx \ln \left( \frac{4K}{|\lambda|} \right) \]

• gives

\[ e^{-\Delta S} = \begin{cases} \ e^{is2\pi - 2s\phi_I} & \text{if } \lambda > 0 = \begin{cases} \left( \frac{|\lambda|}{4K} \right)^{2s} & \text{if } s \in \mathbb{Z} \\ - \left( \frac{|\lambda|}{4K} \right)^{2s} & \text{if } s \in \mathbb{Z} + 1/2 \end{cases} \\ \left( \frac{|\lambda|}{4K} \right)^{2s} & \text{if } \lambda < 0 \end{cases} \]
• Thus we find the matrix element
\[
\langle \downarrow, \uparrow | e^{-\beta H} | \downarrow, \uparrow \rangle = \pm \left( \frac{1}{2} e^{(\frac{|\lambda|}{4K})^{2s} \kappa \beta} - \frac{1}{2} e^{- (\frac{|\lambda|}{4K})^{2s} \kappa \beta} \right) N e^{-\beta E_0}
\]

• where the - sign only applies for the case of half odd integer spin and anti-ferromagnetic coupling.

• An identical analysis gives
\[
\langle \downarrow, \uparrow | e^{-\beta H} | \downarrow, \uparrow \rangle = \langle \uparrow, \downarrow | e^{-\beta H} | \uparrow, \downarrow \rangle = \left( \frac{1}{2} e^{(\frac{|\lambda|}{4K})^{2s} \kappa \beta} + \frac{1}{2} e^{- (\frac{|\lambda|}{4K})^{2s} \kappa \beta} \right) N e^{-\beta E_0}
\]
• For the exact theory, we expect

\[
\langle \downarrow, \uparrow \mid e^{-\beta H} \mid \uparrow, \downarrow \rangle = e^{-\beta(E_0 - \frac{1}{2} \Delta E)} \langle \downarrow, \uparrow \mid E_0 \rangle \langle E_0 \mid \uparrow, \downarrow \rangle + e^{-\beta(E_0 + \frac{1}{2} \Delta E)} \langle \downarrow, \uparrow \mid E_1 \rangle \langle E_1 \mid \uparrow, \downarrow \rangle + \cdots
\]

• and

\[
\langle \downarrow, \uparrow \mid e^{-\beta H} \mid \downarrow, \uparrow \rangle = e^{-\beta(E_0 - \frac{1}{2} \Delta E)} \langle \downarrow, \uparrow \mid E_0 \rangle \langle E_0 \mid \downarrow, \uparrow \rangle + e^{-\beta(E_0 + \frac{1}{2} \Delta E)} \langle \downarrow, \uparrow \mid E_1 \rangle \langle E_1 \mid \downarrow, \uparrow \rangle + \cdots
\]
• which allow us to conclude

\[ \Delta E = 2 \left( \frac{|\lambda|}{4K} \right)^{2s} \kappa \]

• and

\[ |E_0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |\rangle) \]

\[ |E_1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |\rangle) \]

• for antiferromagnetism with integer spin

• but for antiferromagnetic, half odd integer spin

\[ |E_0\rangle = \frac{1}{\sqrt{2}}(|\downarrow, \uparrow\rangle - |\uparrow, \downarrow\rangle) \]

\[ |E_1\rangle = \frac{1}{\sqrt{2}}(|\downarrow, \uparrow\rangle + |\uparrow, \downarrow\rangle) \]
Haldane like spin chain

- We study a periodic chain of spins, in the large spin limit with Hamiltonian:

\[
\hat{H} = -K \sum_{i=1}^{N} S_{i,z}^2 + \lambda \sum_{i=1}^{N} \vec{S}_i \cdot \vec{S}_{i+1}
\]

- The coupling constant \(K\) is assumed to be large, compared to \(\lambda\). This is the opposite limit from what Haldane took in his seminal paper:

\[
H = |J| \sum_n \left[ \vec{S}_n \cdot \vec{S}_{n+1} + \lambda S_n^z S_{n+1}^z + \mu (S_n^z)^2 \right]
\]
• For zero coupling, the ground state is $2^N$ fold degenerate, with each spin being fully up or fully down along the $z$ axis.

• With the added exchange interaction, the spin chain tries to assume a Néel state.

• For an even number of spins, this is possible without frustration, but for an odd number of spins, there must be at least one defect in the Néel order.

• We find the low lying excitations of the even and odd spin cases remarkably different.
The spin coherent states path integral

- We use the path integral to compute transition amplitudes:

\[
\langle \psi | e^{-\beta H} | \chi \rangle = \int \mathcal{D}\{\theta_i, \phi_i\} e^{-S_E}
\]

- Where the Euclidean action is given by:

\[
L_E = is \sum_i \dot{\phi}_i (1 - \cos \theta_i) + K \sum_i \sin^2 \theta_i \\
+ \lambda \sum_i [\sin \theta_i \sin \theta_{i+1} \cos (\phi_i - \phi_{i+1}) + \cos \theta_i \cos \theta_{i+1}]
\]
Spin chain with an even number of sites

- This case admits the Néel states as reasonable approximations to the vacuum.
- However, they can tunnel into each other.
- The classical equation of motion is:

\[ is \frac{d(1 - \cos \theta_i)}{d\tau} = \sin \theta_{i-1} \sin \theta_i \sin(\phi_{i-1} - \phi_i) \]
\[- \sin \theta_i \sin \theta_{i+1} \sin(\phi_i - \phi_{i+1}) \]
• This equation admits a first integral:
\[ \sum_{i} \frac{d(1 - \cos \theta_i)}{d\tau} \Rightarrow \sum_{i} \cos \theta_i = l = 0 \]

• We take the solution:
\[ \theta_{2k} = \pi - \theta \quad \theta_{2k-1} \equiv \theta \]

• Which gives:
\[ L_{E}^{eff} = iS \sum_{k=1}^{N} \dot{\phi}_{k} - iS \cos \theta \sum_{k=1}^{N/2} (\dot{\phi}_{2k-1} - \dot{\phi}_{2k}) \]
\[ + \sum_{i=1}^{N} \left[ K + \lambda [1 + \cos(\phi_i - \phi_{i+1})] \right] \sin^2 \theta \]
• Making the further ansatz: \( \phi_i - \phi_{i+1} = (-1)^{i+1} \phi \)

• This gives the Lagrangian:
\[
L_{E}^{\text{eff}} = isN \dot{\Phi} - \frac{isN}{2} \dot{\phi} \cos \theta + U_{\text{eff}}
\]

• where
\[
U_{\text{eff}} = N[K + \lambda(1 + \cos \phi)] \sin^2 \theta
\]

• Thus the \( N \) spin Hamiltonian reduces down to a single effective spin degree of freedom.

• Conservation of energy gives: \( \partial_\tau U_{\text{eff}} = 0 \)

• which implies:
\[
\cos \phi = - \left( \frac{K}{\lambda} + 1 \right) \ll -1
\]
The solution is \( \phi = \pi + i \phi_I \)
with \( \cosh \phi_I = \left( \frac{K}{\lambda} + 1 \right) \)
And the \( \theta \) equation is:
\[
is \dot{\theta} = -2\lambda \sin \theta \sin \phi = i2\lambda \sin \theta \sinh \phi_I
\]
with solution \( \theta(\tau) = 2 \arctan \left( e^{\omega (\tau - \tau_0)} \right), \omega = (2\lambda/s) \sinh \phi_I \)
This solution is irrelevant, its action is zero.
The action comes from the complex \( \phi = \pi + i \phi_I \)
\[
S_c = S_0 - \frac{isN}{2} \int_0^{\pi+i\phi_I} d\phi \cos \theta|_{\theta=0} - \frac{isN}{2} \int_{\pi+i\phi_I}^{0} d\phi \cos \theta|_{\theta=\pi}
\]
\[
= 0 - isN\pi + Ns\phi_I = -isN\pi + Ns\phi_I
\]
• Thus we need:

\[ \phi_I = \text{arccosh}\left(\frac{K}{\lambda} + 1\right) \approx 2 \ln\left(\frac{K}{\lambda}\right) \]

• This gives

\[ S_c = -isN\pi + Ns \ln\left(\frac{2K}{\lambda}\right) \]

• The energy splitting is then given by:

\[ \Delta = 2D e^{-S_c} = 2D \left(\frac{\lambda}{2K}\right)^{Ns} \cos(sN\pi) \]

• This is negative for \( N=2(2k+1) \) and half odd integer spin, but otherwise positive.
This gives the ground state is symmetric superposition of the two Néel states \(|+\rangle\) for all values except \(N=2(2k+1)\) and half odd integer spins, for which it is the anti-symmetric superposition \(|-\rangle\).

The dependence on the coupling constant \(\lambda^N s\) indicates that the result can be obtained in high orders in perturbation theory.

In the thermodynamic limit, the two Néel states become degenerate showing parity is spontaneously broken.
Odd number of sites

- The previous description is markedly different when one considers an odd number of sites.
- Here the Néel state is frustrated, there is necessarily a defect.
- As the position of the defect is arbitrary, the ground state is $N$ fold degenerate.

\[ | k \rangle = | \uparrow, \downarrow, \uparrow, \downarrow, \uparrow, \ldots, \uparrow, \uparrow, \ldots, \uparrow, \downarrow \rangle \]

\[ k, k+1^{th} \text{ place} \]

\[ \uparrow, \downarrow, \uparrow, \uparrow, \downarrow, \uparrow, \downarrow \rightarrow \uparrow, \downarrow, \uparrow, \downarrow, \uparrow, \uparrow, \downarrow \]
• Tunnelling allows the state \( |k\rangle \) to mix with other such states.

• We tried to find the instanton that does this, but were unsuccessful. In fact flipping the spins at positions \( k+1 \) and \( k+2 \) yields the state \( |k+2\rangle \), but this should occur at order \( \lambda^{2s} \).

• But the interaction at this order contains two terms which can flip the spins:

\[
\left(S_{k+1}^- S_{k+2}^+\right)^{2s} \quad \left(S_{k-1}^+ S_k^-\right)^{2s}
\]

• Generating the transitions:

\[
|k\rangle \rightarrow |k+2\rangle \quad \quad |k\rangle \rightarrow |k-2\rangle
\]
• Thus at this order the degenerate ground states are mixed. To find the linear combination which yields the true ground state we must diagonalize the corresponding matrix of transition amplitudes. It is of the form:

\[
[b_{\mu,\nu}] = C \left( \begin{array}{ccccccc}
0 & 0 & 1 & 0 & \ldots & 1 & 0 \\
0 & 0 & 0 & 1 & \ldots & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & \ldots & 0 \\
\vdots & 1 & 0 & \ddots & \ddots & \ddots & \\
1 & \ldots & \ddots & \ldots & 0 & 0 & 0 \\
0 & 1 & \ldots & 1 & \ldots & 0 & 0 \\
0 & 0 & 1 & \ldots & 1 & \ldots & 0 \\
\end{array} \right)
\]
• Where the coefficient is calculable as:

\[ C = \pm 4K s^2 \left( \frac{\lambda}{2K} \right)^{2s} \]

• The minus sign is for integer spin while the plus sign is for half odd integer spin.

• The matrix is of the circulant type, they can be easily diagonalized using the roots of unity

\[ | \frac{2\pi j}{N} \rangle = (1, \omega_j, \omega_j^2, \cdots, \omega_j^{N-1}) \]

\[ \varepsilon_j = b_{1,1} + b_{1,2}\omega_j + b_{1,3}\omega_j^2 + \cdots + b_{1,N}\omega_j^{N-1} \]

• where \( \omega_j = e^{i\frac{2\pi j}{N}} \) is the \( j^{\text{th}} \), \( N^{\text{th}} \) root of unity.
• As there are only two non-zero components in the first (any) row, we get:

$$\varepsilon_j = C(\omega_j^2 + \omega_j^{N-2}) = C(\omega_j^2 + \omega_j^{-2})$$

$$= 2C \cos \left( \frac{4\pi j}{N} \right).$$

$C < 0$  

$C > 0$
The spectrum is symmetric about $N/2$:

$$\cos \left( \frac{4\pi([N/2]-k)}{N} \right) = \cos \left( \frac{4\pi([N/2]+k+1)}{N} \right)$$

$k = 0, 1, 2, \ldots, [N/2]-1$

- Except, the state at $k=[N/2]$ is not paired.
- For integer spins $C < 0$ this state is the unique ground state.
- For half-odd integer spins $C > 0$ and the ground state is doubly degenerate, in accordance with Kramer’s theorem.
Conclusions

• We have found the low lying states of a simple model of two interacting spins with easy axis anisotropy and and exchange interaction.

• The spin coherent state path integral gives a tunnelling like contribution, which actually corresponds to the 2s order in perturbation theory.

• Applications to real systems might be important: molecular magnets, q-bits, digital memory, ground state of the anti-ferromagnet.
• Even spin periodic chain has a non-degenerate ground state which is the symmetric or the anti-symmetric superposition of the two Néel states depending on the spin and the number of sites. The two superpositions are split in energy by \((\frac{\lambda}{2K})^sN\). The excitation spectrum has a gap, proportional to \(4\lambda\) and corresponding to the creation of a soliton anti-soliton pair. The spin waves are highly gapped due to the large anisotropy.

• Odd spin periodic chain has a gapless spectrum. The chain must contain at least one soliton. The chain with one up-up soliton has total spin \(s\), while the one with a down down soliton has total spin \(-s\), and these two sectors do not mix. As the position of the solitons is arbitrary, each sector is \(N\) fold degenerate. Transitions between the ground states breaks the degeneracy and form a gapless band.