Quintessence, NON SINGULAR COSMOLOGIES and Unified DE/DM

EDUARDO GUENDELMAN,
PHYSICS DEPARTMENT, BEN GURION UNIVERSITY, BEER SHEVA, ISRAEL.
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Subject of this talk: The origin of the universe, if possible 1) avoiding the initial singularity, 2) subsequent inflation and 3) transition to the present very slowly accelerated phase (as compared to the inflationary phase). Do all this while addressing the 4) small cosmological constant problem and the DM/DE question. 1) relates to the consistency of theory while 3) is by now a well established observational fact. 4) is by now a well established observational fact.
WHY INFLATION?

Does not have anything to say about initial singularity, but addresses other problems in modern cosmology:

1) Why the universe is so homogeneous and isotropic, while the standard model (before inflation) told us that the present observed universe consisted of many causally disconnected pieces, the so called horizon problem

2) Why the universe is so nearly spatially flat?

1) is solved by starting from a very small region where everything was causally connected and inflating it.

2) The inflation naturally flattens any starting space.
The cosmological constant Problem

Huge differences in scale between inflationary vacuum energy density and present energy density.....
SUMMARY OF OUR RESULTS

Here we will see how a unified picture of inflation and present dark energy can be consistent with a smooth, non singular origin of the universe, represented by the emergent scenario, presenting an attractive cosmological scenario. For the late universe the DM is considered. This is achieved by considering two non Riemannian measures or volume forms in the action. The motivation is:
The early inflation, although solving many cosmological puzzles, like the horizon and flatness problems, cannot address the initial singularity problem;

There is no explanation for the existence of two periods of exponential expansion with such wildly different scales – the inflationary phase and the present phase of slowly accelerated expansion of the universe.

The best known mechanism for generating a period of accelerated expansion is through the presence of some vacuum energy. In the context of a scalar field theory, vacuum energy density appears naturally when the scalar field acquires an effective potential $U_{\text{eff}}$ which has flat regions so that the scalar field can "slowly roll"

and its kinetic energy can be neglected resulting in an energy-momentum tensor $T_{\mu\nu} \approx g_{\mu\nu} U_{\text{eff}}$.

we will study a unified scenario where both an inflation

and a slowly accelerated phase for the universe can appear naturally from the existence of two flat regions in the effective scalar field potential which we derive systematically from a Lagrangian action principle. Namely, we start with a new kind of globally Weyl-scale invariant gravity-matter action within the first-order (Palatini) approach formulated in terms of two different non-Riemannian volume forms (integration measures).
Alternative spacetime volume-forms (generally-covariant integration measure densities) independent on the Riemannian metric on the pertinent spacetime manifold have profound impact in (field theory) models with general coordinate reparametrization invariance – general relativity and its extensions, strings and (higher-dimensional) membranes.

Among the principal new phenomena are:

- (i) new mechanism of dynamical generation of cosmological constant;
- (ii) new mechanism of dynamical spontaneous breakdown of supersymmetry in supergravity;
- (iii) new type of "quintessential inflation" scenario in cosmology;
In standard generally-covariant theories (with action $S = \int d^Dx \sqrt{-g} \mathcal{L}$) the Riemannian spacetime volume-form, i.e., the integration measure density is given by $\sqrt{-g}$, where $g \equiv \text{det} \|g_{\mu\nu}\|$ is the determinant of the corresponding Riemannian metric $g_{\mu\nu}$.

$\sqrt{-g}$ transforms as scalar density under general coordinate reparametrizations.

There is NO a priori any obstacle to employ instead of $\sqrt{-g}$ another alternative non-Riemannian volume element given by the following non-Riemannian integration measure density:

$$\Phi(B) \equiv \frac{1}{(D-1)!} \varepsilon^{\mu_1 \ldots \mu_D} \partial_{\mu_1} B_{\mu_2 \ldots \mu_D}.$$

Here $B_{\mu_1 \ldots \mu_{D-1}}$ is an auxiliary rank $(D-1)$ antisymmetric tensor.
1. $\Phi(B)$ similarly transforms as scalar density under coordinate reparametrizations.

In particular, $B_{\mu_1...\mu_{D-1}}$ can also be parametrized in terms of $D$ auxiliary scalar fields:

$$B_{\mu_1...\mu_{D-1}} = \frac{1}{D!} \varepsilon^I J_1...J_{D-1} \phi^I \partial_{\mu_1} \phi^{J_1} \ldots \partial_{\mu_{D-1}} \phi^{J_{D-1}},$$

so that:

$$\Phi(B) = \frac{1}{D!} \varepsilon^{\mu_1...\mu_D} \varepsilon^{I_1...I_D} \partial_{\mu_1} \phi^{I_1} \ldots \partial_{\mu_D} \phi^{I_D}.$$ 

This density can be built out of four auxiliary scalar fields $\varphi^i$ ($i = 1, 2, 3, 4$):

$$\Phi(\varphi) = \frac{1}{4!} \varepsilon^{\mu\nu\kappa\lambda} \varepsilon_{ijkl} \partial_{\mu} \varphi^i \partial_{\nu} \varphi^j \partial_{\kappa} \varphi^k \partial_{\lambda} \varphi^l.$$ 

$\Phi(\varphi)$ is a scalar density under general coordinate transformations.
The interpretation of this is an alternative realization of a non Riemannian measure, from a mapping of two spaces. Ideas from where can we get 4 scalars, for example from Cederwall and collaborators, to realize duality, by doubling of space time, adding “twiddle” coordinates which are scalars w/r to the “normal space”. We then can define a “brane “ where the twiddle coordinates are a functions of un-twiddle coordinates and Jacobian from the mapping defines measure of integration,
There is also an infinite dimensional symmetry of the action as long as the measures stay linear in the action, then, for example in the formulation where our

density can be built out of four auxiliary scalar fields $\varphi^i$ ($i = 1, 2, 3, 4$):

$$\Phi(\varphi) = \frac{1}{4!} \epsilon^{\mu\nu\kappa\lambda} \epsilon_{ijkl} \partial_\mu \varphi^i \partial_\nu \varphi^j \partial_\kappa \varphi^k \partial_\lambda \varphi^l.$$  

We can shift each of the 4 scalars by an arbitrary function of the lagrangian that multiplies it. If symmetry is preserved in QM, then also the structure will be preserved by QM corrections.
To illustrate the TMT formalism let us consider the following action:

\[
S = c_1 \int d^D x \Phi(B) \left[ L^{(1)} + \frac{\varepsilon^{\mu_1 \cdots \mu_D}}{(D-1)! \sqrt{-g}} \partial_{\mu_1} H_{\mu_2 \cdots \mu_D} \right] + c_2 \int d^D x \sqrt{-g} L^{(2)}
\]

with the following notations:

- The Lagrangians $L^{(1,2)} = \frac{1}{2\kappa^2} R + L^{(1,2)}_{\text{matter}}$ include both standard Einstein-Hilbert gravity action as well as matter/gauge-field parts. Here $R = g^{\mu\nu} R_{\mu\nu}(\Gamma)$ is the scalar curvature within the first-order (Palatini) formalism and $R_{\mu\nu}(\Gamma)$ is the Ricci tensor in terms of the independent affine connection $\Gamma^\mu_{\lambda\nu}$. 
Varying (2) w.r.t. $H$ and $B$ tensor gauge fields we get:

$$\partial_\mu \left( \frac{\Phi(B)}{\sqrt{-g}} \right) = 0 \rightarrow \frac{\Phi(B)}{\sqrt{-g}} \equiv \chi = \text{const},$$

(3)

$$L^{(1)} + \frac{\varepsilon_{\mu_1\ldots\mu_D}}{(D-1)!\sqrt{-g}} \partial_{\mu_1} H_{\mu_2\ldots\mu_D} = M = \text{const},$$

(4)

Now, varying (2) w.r.t. $g^{\mu\nu}$ and taking into account (3)–(4) we arrive at the following effective Einstein equations (in the first-order formalism):

$$R_{\mu\nu}(\Gamma) - \frac{1}{2} g_{\mu\nu} R + \Lambda_{\text{eff}} g_{\mu\nu} = \kappa^2 T^{\text{eff}}_{\mu\nu},$$

(5)

with effective energy-momentum tensor:

$$T^{\text{eff}}_{\mu\nu} = g_{\mu\nu} L^{\text{eff}}_{\text{matter}} - 2 \frac{\partial L^{\text{eff}}_{\text{matter}}}{\partial g^{\mu\nu}}, \quad L^{\text{eff}}_{\text{matter}} \equiv \frac{1}{c_1\chi + c_2} \left[ c_1 L^{(1)}_{\text{matter}} + c_2 L^{(2)}_{\text{matter}} \right],$$

(6)
and with a *dynamically generated effective cosmological constant* thanks to the non-zero integration constants

\[ \Lambda_{\text{eff}} = \kappa^2 (c_1 \chi + c_2)^{-1} \chi M. \]

Let us now consider modified-measure gravity-matter theories constructed in terms of two different non-Riemannian volume-forms (employing again Palatini formalism, and using units where \( G_{\text{Newton}} = 1/16\pi \)):

\[ S = \int d^4 x \, \Phi_1(A) \left[ R + L^{(1)} \right] + \int d^4 x \, \Phi_2(B) \left[ L^{(2)} + \epsilon R^2 + \frac{\Phi(H)}{\sqrt{-g}} \right]. \]

(25)

- \( \Phi_1(A) \) and \( \Phi_2(B) \) are two independent non-Riemannian volume-forms:
Before going further, let us point out also that the Unimodular theory is a very special case of a Two Measures Theory.

**Generalized unimodular gravity:**

\[
S_{\text{unimod}} = -\frac{1}{16\pi G} \int [\sqrt{g} (R + 2\Lambda) - 2\Lambda \partial_{\mu} T^{\mu}](d^3x)dt.
\]

One of its equations of motion is \( \sqrt{g} = \partial_{\mu} T^{\mu} \), the generalized unimodular condition, with \( g \) given in terms of the auxiliary field \( T^{\mu} \) (with \( T^0 \) having the meaning of time).

\[
\Phi(H) = \frac{1}{3!} \varepsilon^{\mu\nu\kappa\lambda} \partial_{\mu} H_{\nu\kappa\lambda} = \partial_{\mu} T^{\mu}.
\]
\[ S = \int d^4x \Phi_1(A) \left[ R + L^{(1)} \right] + \int d^4x \Phi_2(B) \left[ L^{(2)} + \epsilon R^2 + \frac{\Phi(H)}{\sqrt{-g}} \right] \]

\( \Phi_1(A) \) and \( \Phi_2(B) \) are two independent non-Riemannian volume-forms:

\[ \Phi_1(A) = \frac{1}{3!} \varepsilon^{\mu\nu\kappa\lambda} \partial_\mu A_{\nu\kappa\lambda} \quad , \quad \Phi_2(B) = \frac{1}{3!} \varepsilon^{\mu\nu\kappa\lambda} \partial_\mu B_{\nu\kappa\lambda} \]

\[ \Phi(H) = \frac{1}{3!} \varepsilon^{\mu\nu\kappa\lambda} \partial_\mu H_{\nu\kappa\lambda} \]

- \( L^{(1,2)} \) denote two different Lagrangians of a single scalar matter field
The variation with respect to the $H$ three index potential tells us, that on shell, up to a proportionality constant the second measure is the Riemannian measure (the square root of the determinant of the metric). The 2 part of the action has been used in the past for (i) string, super-strings, branes and super-branes, (ii) modified measure formulations of supergravity.

In this case the analogous of the $H$ field is crucial to implement supersymmetry. In case of the extended objects the proportionality constant between the measure and the Riemannian measure represents the generation of a dynamical tension of the extended object.
$L^{(1,2)}$ denote two different Lagrangians of a single scalar matter field of the form:

\[ L^{(1)} = -\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi), \quad V(\varphi) = f_1 \exp\{-\alpha \varphi\}, \]

\[ L^{(2)} = -\frac{b}{2} e^{-\alpha \varphi} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + U(\varphi), \quad U(\varphi) = f_2 \exp\{-2\alpha \varphi\}, \]

where $\alpha, f_1, f_2$ are dimensionful positive parameters, whereas $b$ is a dimensionless one.

**Global Weyl-scale invariance of the action**

$g_{\mu\nu} \rightarrow \lambda g_{\mu\nu}, \quad \Gamma^\mu_{\nu\lambda} \rightarrow \Gamma^\mu_{\nu\lambda}, \quad \varphi \rightarrow \varphi + \frac{1}{\alpha} \ln \lambda,$

$A_{\mu\nu\kappa} \rightarrow \lambda A_{\mu\nu\kappa}, \quad B_{\mu\nu\kappa} \rightarrow \lambda^2 B_{\mu\nu\kappa}, \quad H_{\mu\nu\kappa} \rightarrow H_{\mu\nu\kappa},$

$\lambda$ (1) with affine connection $\Gamma^\nu_{\nu\lambda}.$

\[ \int d^4 x \sqrt{-g} g^{\mu\nu} \left( \frac{\Phi_1}{\sqrt{-g}} + 2\epsilon \frac{\Phi_2}{\sqrt{-g}} R \right) (\nabla_\kappa \delta \Gamma^\kappa_{\mu\nu} - \nabla_\mu \delta \Gamma^\kappa_{\kappa\nu}) = 0 \]  
(7)
Eqs. of motion w.r.t. affine connection $\Gamma^\mu_{\nu\lambda}$ yield a solution for the latter as a Levi-Civita connection:

$$\Gamma^\mu_{\nu\lambda} = \Gamma^\mu_{\nu\lambda}(\bar{g}) = \frac{1}{2}\bar{g}^{\mu\kappa}(\partial_\nu \bar{g}_{\lambda\kappa} + \partial_\lambda \bar{g}_{\nu\kappa} - \partial_\kappa \bar{g}_{\nu\lambda}) \ ,$$  \hspace{1cm} (30)

w.r.t. to the Weyl-rescaled metric $\bar{g}_{\mu\nu}$:

$$\bar{g}_{\mu\nu} = (\chi_1 + 2\epsilon \chi_2 R)g_{\mu\nu} \ , \ \chi_1 \equiv \frac{\Phi_1(A)}{\sqrt{-g}} \ , \ \chi_2 \equiv \frac{\Phi_2(B)}{\sqrt{-g}} \ .$$  \hspace{1cm} (31)
Variation of the action (25) w.r.t. auxiliary tensor gauge fields $A_{\mu\nu\lambda}$, $B_{\mu\nu\lambda}$ and $H_{\mu\nu\lambda}$ yields the equations:

$$
\partial_\mu \left[ R + L^{(1)} \right] = 0, \quad \partial_\mu \left[ L^{(2)} + \epsilon R^2 + \frac{\Phi(H)}{\sqrt{-g}} \right] = 0, \quad \partial_\mu \left( \frac{\Phi_2(B)}{\sqrt{-g}} \right) = 0,
$$

(32)

whose solutions read:

$$
\frac{\Phi_2(B)}{\sqrt{-g}} \equiv \chi_2 = \text{const}, \quad R + L^{(1)} = -M_1 = \text{const},
$$

$$
L^{(2)} + \epsilon R^2 + \frac{\Phi(H)}{\sqrt{-g}} = -M_2 = \text{const}.
$$

(33)

Here $M_1$ and $M_2$ are arbitrary dimensionful and $\chi_2$ arbitrary dimensionless integration constants.
The first integration constant $\chi_2$ in (33) preserves global Weyl-scale invariance whereas the appearance of the second and third integration constants $M_1, M_2$ signifies dynamical spontaneous breakdown of global Weyl-scale invariance due to the scale non-invariant solutions (second and third ones) in (33).

\[
T_{\mu\nu}^{(1,2)} = g_{\mu\nu} L^{(1,2)} - 2 \frac{\partial}{\partial g_{\mu\nu}} L^{(1,2)}. 
\]

the scale factor $\chi_1$:

\[
\chi_1 = 2\chi_2 \frac{T^{(2)}/4 + M_2}{L^{(1)} - T^{(1)}/2 - M_1}.
\]

where $T^{(1,2)} = g_{\mu\nu} T_{\mu\nu}^{(1,2)}$.

\[
\chi_1 \left[ R_{\mu\nu} + \frac{1}{2} \left( g_{\mu\nu} L^{(1)} - T_{\mu\nu}^{(1)} \right) \right] - \frac{1}{2} \chi_2 \left[ T_{\mu\nu}^{(2)} + g_{\mu\nu} \left( \epsilon R^2 + M_2 \right) - 2 R R_{\mu\nu} \right] = 0, (12)
\]

where $\chi_1$ and $\chi_2$ are defined in (9), and $T_{\mu\nu}^{(1,2)}$ are the energy-momentum tensors of the scalar field Lagrangians with the standard definitions:
of the scalar field Lagrangians with the standard definitions:

$$T^{(1,2)}_{\mu\nu} = g_{\mu\nu} L^{(1,2)} - 2 \frac{\partial}{\partial g_{\mu\nu}} L^{(1,2)} . \quad (13)$$

Taking the trace of Eqs.(12) and using again second relation (11) we solve for the scale factor $\chi_1$:

$$\chi_1 = 2\chi_2 \frac{T^{(2)}/4 + M_2}{L^{(1)} - T^{(1)}/2 - M_1} , \quad (14)$$

where $T^{(1,2)} = g^{\mu\nu} T^{(1,2)}_{\mu\nu}$.

Using second relation (11) Eqs.(12) can be put in the Einstein-like form:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{2} g_{\mu\nu} \left( L^{(1)} + M_1 \right) + \frac{1}{2\Omega} \left( T^{(1)}_{\mu\nu} - g_{\mu\nu} L^{(1)} \right)$$

$$+ \frac{\chi_2}{2\chi_1\Omega} \left[ T^{(2)}_{\mu\nu} + g_{\mu\nu} \left( M_2 + \epsilon (L^{(1)} + M_1)^2 \right) \right] , \quad (15)$$

where:

$$\Omega = 1 - \frac{\chi_2}{\chi_1} 2\epsilon \left( L^{(1)} + M_1 \right) . \quad (16)$$

Let us note that (9), upon taking into account second relation (11) and (16), can be written as:

$$\bar{g}_{\mu\nu} = \chi_1 \Omega g_{\mu\nu} . \quad (17)$$
Now, we can bring Eqs. (15) into the standard form of Einstein equations for the rescaled metric \( \bar{g}_{\mu \nu} \) (17), i.e., the Einstein-frame equations:

\[
R_{\mu \nu} (\bar{g}) - \frac{1}{2} \bar{g}_{\mu \nu} R(\bar{g}) = \frac{1}{2} T_{\mu \nu}^{\text{eff}}
\]  

(18)

with energy-momentum tensor corresponding (according to (13)) to the following effective (Einstein-frame) scalar field Lagrangian:

\[
L_{\text{eff}} = \frac{1}{\chi_1 \Omega} \left\{ L^{(1)} + M_1 + \frac{\chi_2}{\chi_1 \Omega} \left[ L^{(2)} + M_1 + \varepsilon (L^{(1)} + M_1)^2 \right] \right\} .
\]  

(19)

In order to explicitly write \( L_{\text{eff}} \) in terms of the Einstein-frame metric \( \bar{g}_{\mu \nu} \) (17) we use the short-hand notation for the scalar kinetic term:

\[
X \equiv -\frac{1}{2} \bar{g}^{\mu \nu} \partial_\mu \varphi \partial_\nu \varphi
\]  

(20)

and represent \( L^{(1,2)} \) in the form:

\[
L^{(1)} = \chi_1 \Omega X - V , \quad L^{(2)} = \chi_1 \Omega b e^{-\alpha \varphi} X + U ,
\]  

(21)

with \( V \) and \( U \) as in (3)-(4).

From Eqs. (14) and (16), taking into account (21), we find:

\[
\frac{1}{\chi_1 \Omega} = \frac{(V - M_1)}{2 \chi_2 \left[ U + M_2 + \varepsilon (V - M_1)^2 \right]} \left[ 1 - \chi_2 \left( \frac{b e^{-\alpha \varphi}}{V - M_1} - 2 \varepsilon \right) X \right] .
\]  

(22)

Upon substituting expression (22) into (19) we arrive at the explicit form for the Einstein-frame scalar Lagrangian:

\[
L_{\text{eff}} = A(\varphi) X + B(\varphi) X^2 - U_{\text{eff}}(\varphi) ,
\]  

(23)
Performing transition to the Einstein frame yields the following effective scalar Lagrangian of non-canonical “k-essence” (kinetic quintessence) type \( (X \equiv -\frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - \text{scalar kinetic term}) \):

\[
L_{\text{eff}} = A(\varphi) X + B(\varphi) X^2 - U_{\text{eff}}(\varphi) ,
\]

where (recall \( V = f_1 e^{-\alpha \varphi} \) and \( U = f_2 e^{-2\alpha \varphi} \)):

\[
A(\varphi) \equiv 1 + \left[ \frac{1}{2} b e^{-\alpha \varphi} - \epsilon (V - M_1) \right] \frac{V - M_1}{U + M_2 + \epsilon (V - M_1)^2} ,
\]

\[
B(\varphi) \equiv \chi_2 \frac{\epsilon \left[ U + M_2 + (V - M_1) b e^{-\alpha \varphi} \right] - \frac{1}{4} b^2 e^{-2\alpha \varphi}}{U + M_2 + \epsilon (V - M_1)^2} ,
\]

\[
U_{\text{eff}}(\varphi) \equiv \frac{(V - M_1)^2}{4 \chi_2 \left[ U + M_2 + \epsilon (V - M_1)^2 \right]} .
\]
Most remarkable feature of the effective scalar potential $U_{\text{eff}}(\varphi)$ (37) – two infinitely large flat regions:

- **(-) flat region** – for large negative values of $\varphi$:
  \[
  U_{\text{eff}}(\varphi) \sim U_{(-)} = \frac{f_1^2/f_2}{4\chi_2(1 + \epsilon f_1^2/f_2)}, \tag{38}
  \]

- **(+) flat region** – for large positive values of $\varphi$:
  \[
  U_{\text{eff}}(\varphi) \sim U_{(+) = \frac{M_1^2/M_2}{4\chi_2(1 + \epsilon M_1^2/M_2)}}, \tag{39}
  \]
Shape of the effective scalar potential $U_{\text{eff}}(\varphi)$ (26) for $M_1 < 0$. 
Interesting feature of the effective potential in the case no $R^2$ terms are introduced, which is that the bump showed in the picture raises above the first flat region (relevant for the early universe) as much as the second flat region (relevant for the present universe) is above zero! . There is a hint of the present universe which appears in the early universe.

For the other sign, we get a shape similar to that of the Starobinsky model, which provides a good description of inflation, but now, the new thing is the additional flat region that can take care of the present dark energy!, see:
Shape of the effective scalar potential $U_{\text{eff}}(\varphi)$ (26) for $M_1 > 0$. 
This shape is of interest in that it addresses in a sense the cosmological constant problem, the old cosmological constant, due to the fact that we get a perfect square in the numerator of the effective potential.
3 Flat Regions of the Effective Scalar Potential

Depending on the sign of the integration constant $M_1$ we obtain two types of shapes for the effective scalar potential $U_{\text{eff}}(\varphi)$ (26) depicted on Fig.1. and Fig.2.

The crucial feature of $U_{\text{eff}}(\varphi)$ is the presence of two very large flat regions – for negative and positive values of the scalar field $\varphi$. For large negative values of $\varphi$ we have for the effective potential and the coefficient functions in the Einstein-frame scalar Lagrangian (23)-(26):

$$U_{\text{eff}}(\varphi) \simeq U_{(-)} = \frac{f_1^2/f_2}{4\chi_2(1 + \epsilon f_1^2/f_2)}, \quad (27)$$

$$A(\varphi) \simeq A_{(-)} = \frac{1 + \frac{1}{2}bf_1/f_2}{1 + \epsilon f_1^2/f_2}, \quad B(\varphi) \simeq B_{(-)} = -\chi_2 \frac{b^2/4f_2 - \epsilon(1 + bf_1/f_2)}{1 + \epsilon f_1^2/f_2}. \quad (28)$$

In the second flat region for large positive $\varphi$:

$$U_{\text{eff}}(\varphi) \simeq U_{(+)0} = \frac{M_1^2/M_2}{4\chi_2(1 + \epsilon M_1^2/M_2)}, \quad (29)$$

$$A(\varphi) \simeq A_{(+)0} = \frac{M_2}{M_2 + \epsilon M_1^2}, \quad B(\varphi) \simeq B_{(+)0} = \epsilon \chi_2 \frac{M_2}{M_2 + \epsilon M_1^2}. \quad (30)$$
From the expression for \( U_{\text{eff}}(\varphi) \) (26) and the figures 1 and 2 we see that now we have an explicit realization of quintessential inflation scenario. The flat regions (27)-(28) and (29)-(30) correspond to the evolution of the early and the late universe, respectively, provided we choose the ratio of the coupling constants in the original scalar potentials versus the ratio of the scale-symmetry breaking integration constants to obey:

\[
\frac{f_1^2/f_2}{1 + \epsilon f_1^2/f_2} \gg \frac{M_1^2/M_2}{1 + \epsilon M_1^2/M_2},
\]

which makes the vacuum energy density of the early universe \( U_{(-)} \) much bigger than that of the late universe \( U_{(+)} \) (cf. (27), (29)). The inequality (31) is equivalent to the requirements:

\[
\frac{f_1^2}{f_2} \gg \frac{M_1^2}{M_2}, \quad |\epsilon| \frac{M_1^2}{M_2} \ll 1.
\]

In particular, if we choose the scales of the scale symmetry breaking integration constants \( M_1 \sim M_{\text{EW}}^4 \) and \( M_2 \sim M_{\text{Pl}}^4 \), where \( M_{\text{EW}}, M_{\text{Pl}} \) are the electroweak and Plank scales, respectively, we are then naturally led to a very small vacuum energy density \( U_{(+)} \sim M_1^2/M_2 \) of the order \( M_{\text{EW}}^8/M_{\text{Pl}}^4 \sim 10^{-120} M_{\text{Pl}}^4 \), which is the right order of magnitude for the present eophe's vacuum energy density as already recognized in Ref.[27]. On the other hand, if we take the order of magnitude of the coupling constants in the effective potential \( f_1 \sim f_2 \sim (10^{-2} M_{\text{Pl}})^4 \), then together with the above choice of order of magnitudes for \( M_{1,2} \) the inequalities (32) will be satisfied as well and the order of magnitude of the vacuum energy density of the early universe \( U_{(-)} \) (27) becomes \( U_{(-)} \sim f_1^2/f_2 \sim 10^{-8} M_{\text{Pl}}^4 \) which conforms to the BICEP2 data [5] implying the energy scale of inflation of order \( 10^{-2} M_{\text{Pl}} \).
WE OBTAIN THE SEE-SAW FORMULA FOR PRESENT VACUUM ENERGY DENSITY

\[ U_{(+)} \sim \frac{\tilde{M}_1^2}{M_2} \text{ of the order } M_{EW}^8/M_{Pl}^4 \sim 10^{-120} M_{Pl}^4 \]
It is interesting to notice that although the two constants of integration individually violate scale invariance, the combination which appears in their contribution to the asymptotic value of the effective potential in one of the flat regions

\[ \frac{M_1^2}{M_2} \]

is scale invariant
MULTI UNIVERSE INTERPRETATION OF INTEGRATION CONSTANTS?

• MAY BE, BUT NOTICE, INTEGRATION CONSTANTS IRRELEVANT FOR EARLY UNIVERSE!

• SO EARLY UNIVERSE IS

..........................................................
MULTI UNIVERSAL

..........................................................
Before proceeding to the derivation of the non-singular "emergent universe" solution describing an initial phase of the universe evolution preceding the inflationary phase, let us briefly sketch how the present non-Riemannian-measure-modified gravity-matter theory meets the conditions for the validity of the "slow-roll" approximation [6] when $\varphi$ evolves on the flat region of the effective potential corresponding to the early universe (27)-(28).

To this end let us recall the standard Friedman-Lemaître-Robertson-Walker space-time metric [26]:

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2)\right]$$  \hspace{1cm} (33)

and the associated Friedman equations (recall the presently used units $G_{\text{Newton}} = 1/16\pi$):

$$\frac{\ddot{a}}{a} = -\frac{1}{12}(\rho + 3p) \quad , \quad H^2 + \frac{K}{a^2} = \frac{1}{6}\rho \quad , \quad H \equiv \frac{\dot{a}}{a} ,$$  \hspace{1cm} (34)

describing the universe's evolution. Here:

$$\rho = \frac{1}{2}A(\varphi)\dot{\varphi}^2 + \frac{3}{4}B(\varphi)\dot{\varphi}^4 + U_{\text{eff}}(\varphi) ,$$  \hspace{1cm} (35)

$$p = \frac{1}{2}A(\varphi)\dot{\varphi}^2 + \frac{1}{4}B(\varphi)\dot{\varphi}^4 - U_{\text{eff}}(\varphi)$$  \hspace{1cm} (36)

are the energy density and pressure of the scalar field $\varphi = \varphi(t)$. Henceforth the dots indicate derivatives with respect to the time $t$. 
Let us now consider the standard “slow-roll” parameters [7]:

\[ \varepsilon \equiv -\frac{\dot{H}}{H^2}, \quad \eta \equiv -\frac{\ddot{\varphi}}{H \dot{\varphi}}, \]  \tag{37} 

where \( \varepsilon \) measures the ratio of the scalar field kinetic energy relative to its total energy density and \( \eta \) measures the ratio of the fields acceleration relative to the “friction” (\( \sim 3H \dot{\varphi} \)) term in the pertinent scalar field equations of motion:

\[ \ddot{\varphi} (A + 3B \dot{\varphi}^2) + 3H \dot{\varphi} (A + B \dot{\varphi}^2) + U'_\text{eff} + \frac{1}{2} A' \dot{\varphi}^2 + \frac{3}{4} B' \dot{\varphi}^4 = 0, \]  \tag{38} 

with primes indicating derivatives w.r.t. \( \varphi \).

In the slow-roll approximation one ignores the terms with \( \ddot{\varphi}, \dot{\varphi}^2, \dot{\varphi}^3, \dot{\varphi}^4 \) so that the \( \varphi \)-equation of motion (38) and the second Friedman Eq.(34) reduce to:

\[ 3AH \dot{\varphi} + U'_\text{eff} = 0, \quad H^2 = \frac{1}{6} U'\text{eff}. \]  \tag{39} 

Now, using the fact that \( \varphi \) evolves on a flat region of \( U'_\text{eff} \) we deduce that \( H \equiv \dot{a}/a \simeq \text{const} \), so that \( a(t) \) grows exponentially with time and, thus, in the second Eq.(39) the spatial curvature term \( K/a^2 \) is ignored. Consistency of the slow-roll approximation implies for the slow-roll parameters (37), taking into account (39), the following inequalities:

\[ \varepsilon \simeq \frac{1}{A} \left( \frac{U'_\text{eff}}{U'\text{eff}} \right)^2 \ll 1, \quad \eta \simeq \frac{2}{A} \frac{U''\text{eff}}{U'\text{eff}} - \varepsilon - \frac{2A'}{A^{3/2}} \sqrt{\varepsilon} \rightarrow \frac{2}{A} \frac{U''\text{eff}}{U'\text{eff}} \ll 1. \]  \tag{40}
Since now $\varphi$ evolves on the flat region of $U_{\text{eff}}$ for large negative values (27), the Lagrangian coefficient function $A(\varphi) \simeq A_{(-)}$ as in (28) and the gradient of the effective scalar potential is:

$$U'_{\text{eff}} \simeq -\frac{\alpha f_1 M_1 e^{\alpha \varphi}}{2 \chi_2 f_2 (1 + \epsilon f_1^2/f_2)^2},$$  

(41)

which yields for the slow-roll parameter $\varepsilon$ (40):

$$\varepsilon \simeq \frac{4 \alpha^2 M_1^2 e^{2 \alpha \varphi}}{f_1^2 (1 + b f_1/2 f_2)(1 + \epsilon f_1^2/f_2)} \ll 1 \quad \text{for large negative } \varphi.$$  

(42)

Similarly, for the second slow-roll parameter we have:

$$\left| \frac{2}{A} \frac{U''_{\text{eff}}}{U_{\text{eff}}} \right| \simeq \frac{4 \alpha^2 M_1 e^{\alpha \varphi}}{f_1 (1 + b f_1/2 f_2)} \ll 1 \quad \text{for large negative } \varphi.$$  

(43)
The value of $\varphi$ at the end of the slow-roll regime $\varphi_{\text{end}}$ is determined from the condition $\varepsilon \sim 1$ which through (42) yields:

$$e^{-2\alpha \varphi_{\text{end}}} \sim \frac{4\alpha^2 M_1^2}{f_1^2 (1 + bf_1/2f_2)(1 + \epsilon f_1^2/f_2)}$$

(44)

The amount of inflation when $\varphi$ evolves from some initial value $\varphi_{\text{in}}$ to the end-point of slow-roll inflation $\varphi_{\text{end}}$ is determined through the expression for the $e$-foldings $N$ []:

$$N = \int_{\varphi_{\text{in}}}^{\varphi_{\text{end}}} H \, dt = \int_{\varphi_{\text{in}}}^{\varphi_{\text{end}}} \frac{H}{\dot{\varphi}} \, d\varphi \sim -\int_{\varphi_{\text{in}}}^{\varphi_{\text{end}}} \frac{3H^2}{U_{\text{eff}}} \, d\varphi \sim -\int_{\varphi_{\text{in}}}^{\varphi_{\text{end}}} \frac{A U_{\text{eff}}}{2U_{\text{eff}}} \, d\varphi ,$$

(45)

where Eqs.(39) are used. Substituting (27), (28) and (41) into (45) yields an expression for $N$ which together with (44) allows for the determination of $\varphi_{\text{in}}$:

$$N \sim \frac{f_1 (1 + bf_1/f_2)}{4\alpha^2 M_1} \left( e^{-\alpha \varphi_{\text{in}}} - e^{-\alpha \varphi_{\text{end}}} \right).$$

(46)
4 Non-Singular Emergent Universe Solution

We will now show that under appropriate restrictions on the parameters there exist an epoch preceding the inflationary phase. Namely, we derive an explicit cosmological solution of the Einstein-frame system with effective scalar field Lagrangian (23)-(26) describing a non-singular "emergent universe" [15] when the scalar field evolves on the first flat region for large negative $\varphi$ (27). For previous studies of "emergent universe" scenarios within the context of the less general modified-measure gravity-matter theories with one non-Riemannian and one standard Riemannian integration measures, see Ref.[21].

Emergent universe is defined through the standard Friedman-Lemaitre-Robertso-Walker space-time metric (33) as a solution of (34) subject to the condition on the Hubble parameter $H$:

$$H = 0 \quad \rightarrow \quad a(t) = a_0 = \text{const}, \quad \rho + 3p = 0, \quad \frac{K}{a_0^2} = \frac{1}{6} \rho \ (= \text{const}), \quad (47)$$

with $\rho$ and $p$ as in (35)-(36):

The emergent universe condition (47) implies that the $\varphi$-velocity $\dot{\varphi} = \dot{\varphi}_0$ is time-independent and satisfies the bi-quadratic algebraic equation:

$$\frac{3}{2} B_{(-)} \dot{\varphi}_0^4 + 2A_{(-)} \dot{\varphi}_0^2 - 2U_{(-)} = 0 \quad (48)$$

(with notations as in (27)-(28)), whose solution read:

$$\dot{\varphi}_0^2 = -\frac{2}{3B_{(-)}} \left[ A_{(-)} \mp \sqrt{A_{(-)}^2 + 3B_{(-)} U_{(-)}} \right]. \quad (49)$$
To analyze stability of the present emergent universe solution:

\[
a_0^2 = \frac{6k}{\rho_0}, \quad \rho_0 = \frac{1}{2} A_{(-)} \ddot{\varphi}_0^2 + \frac{3}{4} B_{(-)} \dot{\varphi}_0^4 + U_{(-)}, \tag{50}
\]

with \( \ddot{\varphi}_0^2 \) as in (49), we perturb Friedman Eqs. (34) and the expressions for \( \rho, p \)
(35)-(36) w.r.t. \( a(t) = a_0 + \delta a(t) \) and \( \dot{\varphi}(t) = \dot{\varphi}_0 + \delta \dot{\varphi}(t) \), but keep the effective
potential on the flat region \( U_{\text{eff}} = U_{(-)} \):

\[
\frac{\delta \ddot{a}}{a_0} + \frac{1}{12} (\delta \rho + 3 \delta p), \quad \delta \rho = -\frac{2 \rho_0}{a_0} \delta a, \tag{51}
\]

\[
\delta \rho = \left( A_{(-)} \ddot{\varphi}_0 + 3 B_{(-)} \dot{\varphi}_0^3 \right) \delta \dot{\varphi} = -\frac{2 \rho_0}{a_0} \delta a, \quad \delta p = \left( A_{(-)} \ddot{\varphi}_0 + B_{(-)} \dot{\varphi}_0^3 \right) \delta \dot{\varphi}. \tag{52}
\]

From the first Eq. (52) expressing \( \delta \dot{\varphi} \) as function of \( \delta a \) and substituting into the
first Eq. (51) we get a harmonic oscillator type equation for \( \delta a \):

\[
\delta \ddot{a} + \omega^2 \delta a = 0, \quad \omega^2 = \frac{2}{3} \rho_0 \frac{\pm \sqrt{A_{(-)}^2 + 3B_{(-)}U_{(-)}}}{A + 2 \sqrt{A_{(-)}^2 + 3B_{(-)}U_{(-)}}}, \tag{53}
\]

where:

\[
\rho_0 = \frac{1}{2} \dot{\varphi}_0^2 \left[ A_{(-)} + 2 \sqrt{A_{(-)}^2 + 3B_{(-)}U_{(-)}} \right], \tag{54}
\]
with $\dot{\varphi}_0^2$ from (49). Thus, for existence and stability of the emergent universe solution we have to choose the upper signs in (49), (53) and we need the conditions:

$$A_{(-)}^2 + 3B_{(-)} U_{(-)} > 0 \ , \ A_{(-)} - 2\sqrt{A_{(-)}^2 + 3B_{(-)} U_{(-)}} > 0 .$$  (55)

The latter yield the following constraint on the coupling parameters:

$$\max\left\{-2, -8(1 + 3\epsilon f_1^2 / f_2) \left[1 - \sqrt{1 - \frac{1}{4(1 + 3\epsilon f_1^2 / f_2)}}\right]\right\} < b \frac{f_1}{f_2} < -1 ,$$  (56)

in particular, implying that $b < 0$. The latter means that both terms in the original matter Lagrangian $L^{(2)}$ (4) appearing multiplied by the second non-Riemannian integration measure density $\Phi_2$ (2) must be taken with “wrong” signs in order to have a consistent physical Einstein-frame theory (23)-(25) possessing a non-singular emergent universe solution.

For $\epsilon > 0$, since the ratio $\frac{f_1^2}{f_2^2}$ proportional to the height of the first flat region of the effective scalar potential, i.e., the vacuum energy density in the early universe, must be large (cf. (31)), we find that the lower end of the interval in (56) is very close to the upper end, i.e., $b \frac{f_1}{f_2} \simeq -1$. 
The problem of the transition from the Emergent Phase to Inflation

Dynamical systems analysis shows numerically the existence of the transition of the different phases discussed here, non singular emergent universe, followed by inflation, followed by a slowly accelerated phase (today)

The transition from emergent universe to slow roll inflation is interesting
Generalizing the model to include a curvaton field for re-heating

\[ L^{(1)} = -\frac{1}{2} g^{\mu \nu} \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} g^{\mu \nu} \partial_\mu \sigma \partial_\nu \sigma - \frac{\mu^2 \sigma^2}{2} \exp\{-\alpha \varphi\} - V(\varphi), \quad V(\varphi) = f_1 \exp\{-\alpha \varphi\}, \]

\[ L^{(2)} = -\frac{b}{2} e^{-\alpha \varphi} g^{\mu \nu} \partial_\mu \varphi \partial_\nu \varphi + U(\varphi), \quad U(\varphi) = f_2 \exp\{-2\alpha \varphi\}. \]
In the present paper we have constructed a new kind of gravity-matter theory defined in terms of two different non-Riemannian volume-forms (generally covariant integration measure densities) on the space-time manifold, where the Einstein-Hilbert term $R$, its square $R^2$, the kinetic and the potential terms in the pertinent cosmological scalar field (a “dilaton”) couple to each of the non-Riemannian integration measures in a manifestly globally Weyl-scale invariant form. The principal results are as follows:

- Dynamical spontaneous symmetry breaking of the global Weyl-scale invariance.
- In the physical Einstein frame we obtain an effective scalar field potential with two flat regions – one corresponding to the early universe evolution and a second one for the present slowly accelerating phase of the universe.
- The flat region of the effective scalar potential appropriate for describing the early universe allows for the existence of a non-singular “emergent” type beginning of the universe’ evolution. This “emergent” phase is followed by the inflationary phase, which in turn is followed by a period, where the scalar field drops from its high energy density state to the present slowly accelerating phase of the universe.

The flatness of the effective scalar potential in the high energy density region makes the slow rolling inflation regime possible.

The presence of the emergent universe’ phase preceeding the inflationary phase has observable consequences for the low CMB multipoles as has been recently shown in Ref.[29]. Therefore, a full analysis of the CMB results in the context of the present model should involve not only the classical “slow-roll” formalism, but also the “super-inflation” one, which describes the transition from the emergent universe to the inflationary phase.
CONSTRANTS has being performed. Consistent with Planck not BICEP2

<table>
<thead>
<tr>
<th>Phase</th>
<th>Constraint from</th>
<th>Constraint on</th>
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<tr>
<td>Dark energy dominated</td>
<td>vacuum energy density Eq.(34)</td>
<td>$\frac{M^2}{M_P^2} \approx 10^{-120} M_P^4$</td>
</tr>
<tr>
<td>Inflation (using also emergent)</td>
<td></td>
<td></td>
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<tr>
<td></td>
<td>Eq.(35)</td>
<td>$\frac{l_2^2}{f_2} \sim 10^{-8} M_P^4$</td>
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<tr>
<td></td>
<td>$P_S \approx 2.4 \times 10^{-9}$ and $N_* = 60$</td>
<td>$58 \times 10^{-6} \lesssim x_2 \lesssim 74 \times 10^{-3}$</td>
</tr>
<tr>
<td></td>
<td>$n_s = 0.96$ and $N_* = 60$</td>
<td>$3.6 \times 10^{-8} \lesssim f_1 \lesssim 7.7 \times 10^{-8}$</td>
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<tr>
<td></td>
<td>consistency relation $n_s = n_s(r)$</td>
<td>$0 \lesssim \alpha \lesssim 0.2$</td>
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<td>Non-singular emergent</td>
<td>upper end of the interval in Eq. (69)</td>
<td>$b \frac{f_2}{l_2} \simeq -1$</td>
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Table 1. Results for the constraints on the parameters in our model.
Fig. 3 Evolution of the tensor-scalar ratio $r$ versus the scalar spectrum index $n_s$, for three different values of the parameter $\alpha$. The dashed, dotted, and solid lines are for the values of $\alpha = 0.2$, $\alpha = 10^{-2}$ and $\alpha = 10^{-20}$, respectively. Also, in this plot we have taken the values $f_1 = 2 \times 10^{-8}$, $f_2 = 10^{-8}$, $c = 1$, $b = -0.52$ and $M_p = \sqrt{2}$. 
Let us consider the following simple particular case of a non-conventional gravity-scalar-field action – a member of the general class of the "modified-measure" gravity-matter theories (for simplicity we use units with the Newton constant $G_N = 1/16\pi$):

$$S = \int d^4x \sqrt{-g} R + \int d^4x (\sqrt{-g} + \Phi(C)) L(u,Y) .$$

(6)

Here $R$ denotes the standard Riemannian scalar curvature for the pertinent Riemannian metric $g_{\mu\nu}$. In the second term in (6) –
the scalar field Lagrangian is coupled \textit{symmetrically} to two mutually independent spacetime volume-forms – the standard Riemannian $\sqrt{-g}$ and to an alternative non-Riemannian one:

$$\Phi(C) = \frac{1}{3!} \epsilon^\mu\nu\kappa\lambda \partial_\mu C_{\nu\kappa\lambda}$$ \hspace{1cm} (7)$$

$L(u, Y)$ is general-coordinate invariant Lagrangian of a single scalar field $u(x)$, the simplest example being:

$$L(u, Y) = Y - V(u) \hspace{1cm} Y \equiv -\frac{1}{2} g^{\mu\nu} \partial_\mu u \partial_\nu u$$ \hspace{1cm} (8)$$

As it will become clear below, the final result about the unification of dark energy and dark matter resulting from an underlying hidden Noether symmetry of the scalar field action (second term in (6)) does \textit{not} depend on the detailed form of $L(u, Y)$ which could be of an arbitrary generic “k-essence” form.
A crucial new property – we obtain *dynamical constraint* on \( L(u, Y) \) as a result of the equations of motion w.r.t. “measure” huge field \( C_{\mu\nu\lambda} \):

\[
\partial_\mu L(u, Y) = 0 \quad \longrightarrow \quad L(u, Y) = -2M_0 = \text{const} , \quad (10)
\]

In particular, for standard \( L(u, Y) \) \((8)\) \( Y = V(u) - 2M_0 \), where \( M_0 \) arbitrary integration constant. The factor 2 in front of \( M_0 \) is for later convenience in view of its interpretation as a *dynamically generated cosmological constant*.

Indeed, taking into account \((10)\) the energy-momentum tensor becomes:

\[
T_{\mu\nu} = -2g_{\mu\nu}M_0 + \left(1 + \frac{\Phi(C)}{\sqrt{-g}}\right) \frac{\partial L}{\partial Y} \partial_\mu u \partial_\nu u \quad , \quad \nabla^\nu T_{\mu\nu} = 0 . \quad (11)
\]
symmetry of the scalar field action due to the presence of the non-Riemannian volume element $\Phi(C)$:

$$
\delta_\epsilon u = \epsilon \sqrt{Y} \quad , \quad \delta_\epsilon g_{\mu\nu} = 0 \quad , \quad \delta_\epsilon C^\mu = -\epsilon \frac{1}{2\sqrt{Y}} g^{\mu\nu} \partial_\nu u (\Phi(C) + \sqrt{-g})
$$

where $C^\mu \equiv \frac{1}{3!} \epsilon^{\mu\nu\kappa\lambda} C_{\nu\kappa\lambda}$.

Under (12) the action (6) transforms as:

$$
\delta_\epsilon S = \int d^4 x \partial_\mu \left( L(u, Y) \delta_\epsilon C^\mu \right).
$$

Then, standard Noether procedure yields a conserved current:

$$
\nabla_\mu J^\mu = 0 \quad , \quad J^\mu \equiv -\left( 1 + \frac{\Phi(C)}{\sqrt{-g}} \right) \sqrt{2Y} g^{\mu\nu} \partial_\nu u \frac{\partial L}{\partial Y}.
$$

(13)

$T_{\mu\nu}$ (11) and $J^\mu$ (13) can be cast into a relativistic hydrodynamical form (taking into account (10)):
\[ T_{\mu\nu} = -2M_0 g_{\mu\nu} + \rho_0 u_{\mu} u_{\nu} \quad , \quad J^{\mu} = \rho_0 u^{\mu} , \] (14)

where:

\[ \rho_0 \equiv \left( 1 + \frac{\Phi(C)}{\sqrt{-g}} \right) 2Y \frac{\partial L}{\partial Y} , \quad u_{\mu} \equiv -\frac{\partial u}{\sqrt{2Y}} , \quad u^\mu u_\mu = -1 . \] (15)

For the pressure \( p \) and energy density \( \rho \) we have accordingly (with \( \rho_0 \) as in (15)):

\[ p = -2M_0 = \text{const} \quad , \quad \rho = \rho_0 - p = 2M_0 + \left( 1 + \frac{\Phi(C)}{\sqrt{-g}} \right) 2Y \frac{\partial L}{\partial Y} , \] (16)

Because of the constant pressure \( (p = -2M_0) \) \( \nabla^\nu T_{\mu\nu} = 0 \) implies both hidden Noether symmetry current \( J^{\mu} = \rho_0 u^{\mu} \) conservation, as well as geodesic fluid motion:

\[ \nabla_\mu (\rho_0 u^{\mu}) = 0 \quad , \quad u_\nu \nabla^\nu u_\mu = 0 . \] (17)
Therefore, \( T_{\mu\nu} = -2M_0 g_{\mu\nu} + \rho_0 u_\mu u_\nu \) represents an exact sum of two contributions of the two dark species:

\[
p = p_{\text{DE}} + p_{\text{DM}} \quad , \quad \rho = \rho_{\text{DE}} + \rho_{\text{DM}} \quad (18)
\]

\[
p_{\text{DE}} = -2M_0 \quad , \quad \rho_{\text{DE}} = 2M_0 \quad ; \quad p_{\text{DM}} = 0 \quad , \quad \rho_{\text{DM}} = \rho_0 \quad , \quad (19)
\]

i.e., the dark matter component is a dust fluid flowing along geodesics. This is explicit unification of dark energy and dark matter originating from the dynamics of a single scalar field - the “darkon” \( u \).

Let us now consider a combination of the both models above (20) and (6) – gravity coupled to both “inflaton” and “darkon” scalar fields within the non-Riemannian volume-form formalism:

\[
S' = \int d^4x (\sqrt{-g + \Phi(C)})L(u, Y) + \int d^4x \Phi(A) \left[ g^{\mu\nu} R_{\mu\nu}(\Gamma) + L_1(\varphi, X) \right] + \int d^4x \Phi(B) \left[ L_2(\varphi, X) + \frac{\Phi(H)}{\sqrt{-g}} \right] \quad \text{with the same notations}
\]
Following the same steps as above, we derive from (38) the physical **Einstein frame** theory w.r.t. Weyl-rescaled Einstein-frame metric $\bar{g}_{\mu\nu}$ (26) and with an additional “darkon” field redefinition $u \to \tilde{u}$:

$$\frac{\partial \tilde{u}}{\partial u} = (V(u) - 2M_0)^{-\frac{1}{2}} ; \quad Y \to \tilde{Y} = -\frac{1}{2} \bar{g}^{\mu\nu} \partial_{\mu} \tilde{u} \partial_{\nu} \tilde{u} ,$$  \hspace{1cm} (39)

$$S = \int d^4x \sqrt{-\bar{g}} \left[ R(\bar{g}) + L_{\text{eff}}(\varphi, \bar{X}, \tilde{Y}) \right] ,$$  \hspace{1cm} (40)

where (recall $\bar{X} = -\frac{1}{2} \bar{g}^{\mu\nu} \partial_{\mu} \varphi \partial_{\nu} \varphi$):

$$L_{\text{eff}}(\varphi, \bar{X}, \tilde{Y}) = \bar{X} - \tilde{Y} \left( V(\varphi) - M_1 - \chi_2 b e^{-\alpha \varphi} \bar{X} \right) + \tilde{Y}^2 \left[ \chi_2 (U(\varphi) + M_2) - 2M_0 \right]$$  \hspace{1cm} (41)

which is again of a generalized “k-essence” form.

$M_0$ and $M_1, M_2, \chi_2$ are the same integration constants as in (10) and (28), respectively.
The action (40)-(41) possesses an obvious Noether symmetry under the shift $\tilde{u} \to \tilde{u} + \text{const}$ with current conservation:

$$\partial_\mu \left( \sqrt{-\tilde{g}} \tilde{g}^{\mu\nu} \partial_\nu \tilde{u} \frac{\partial L_{\text{eff}}}{\partial \tilde{Y}} \right) \bigg| = 0 ,$$

(42)

which is Einstein-frame counterpart of the original $g_{\mu\nu}$-frame "dust" dark matter density conservation (13).

To study cosmological implications of (38) we perform a FLRW reduction to the class of FLRW metrics:

$$ds^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu = -N^2(t) dt^2 + a^2(t) d\vec{x}.d\vec{x}$$

(43)

and taking the "inflaton" and "darkon" to be time-dependent only:

$$\bar{X} = \frac{1}{2} \dot{\varphi}^2 , \quad \bar{Y} = \frac{1}{2} w^2 , \quad w \equiv \frac{d\tilde{u}}{dt} .$$

(44)

Upon variation w.r.t. "lapse" $N(t)$ we take the gauge $N = 1$. 
Now, the FLRW reduction of the “darkon” $\tilde{\nu}$-eqs. of motion \((42)\) yields a cubic algebraic eq. for its velocity \(w\):

$$
\left[\chi_2(U(\varphi)+M_2)-2M_0\right]w^3-w\left(V(\varphi)-M_1-\chi_2be^{-\alpha\varphi}{1\over 2} \dot{\varphi}^2\right)-{c_0\over a^3} = 0 ,
$$

\((45)\)

where \(c_0\) is an integration constant – the conserved Noether charge of \((42)\) (“dust” dark matter particle number).

Eqs. motion w.r.t. \(N(t)\) and \(a(t)\) (1st and 2nd Friedmann eqs.):

$$
\frac{\dot{a}^2}{a^2} = \frac{1}{6} \rho , \quad \frac{\ddot{a}}{a} = -\frac{1}{12} \left(\rho + 3p\right) ,
$$

\((46)\)

where the energy density \(\rho\) and pressure \(p\) are:

$$
\rho = \frac{1}{2} \dot{\varphi}^2 \left(1 + \frac{3}{4}\chi_2be^{-\alpha\varphi}w^2\right) + \frac{w^2}{4} \left(V(\varphi) - M_1\right) + \frac{3}{4} \frac{c_0}{a^3} w ,
$$

\((47)\)

$$
p = \frac{1}{2} \dot{\varphi}^2 \left(1 + \frac{1}{4}\chi_2be^{-\alpha\varphi}w^2\right) - \frac{w^2}{4} \left(V(\varphi) - M_1\right) + \frac{1}{4} \frac{c_0}{a^3} w .
$$

\((48)\)
Finally, eqs. motion w.r.t. “inflaton” $\varphi$ read:

$$0 = \frac{d}{dt} \left[ a^3 \dot{\varphi} \left( 1 + \frac{x_2}{2} be^{-\alpha\varphi} w^2 \right) \right] + \alpha \frac{x_2 U(\varphi) c_0 w}{x_2 (U(\varphi) + M_2) - 2M_0} + \alpha a^3 w^2 \left\{ \frac{\dot{\varphi}^2}{4} \chi_2 be^{-\alpha\varphi} - \frac{1}{2} V(\varphi) + \frac{x_2 U(\varphi)}{2 \left[ x_2 (U(\varphi) + M_2) - 2M_0 \right]} \left[ V(\varphi) - M_1 - \chi_2 be^{-\alpha\varphi} \dot{\varphi}^2 / 2 \right] \right\}.$$ 

First consider the (+) flat region (34) of the inflaton potential for large positive values of $\varphi$ corresponding to the “late” (nowadays) universe:

$$w = \left[ \frac{|M_1|}{\chi_2 M_2 - 2M_0} \right]^{1/2} + \frac{1}{2|M_1|} \frac{c_0}{a^3} + O\left( \frac{c_0^2}{a^6} \right), \quad (50)$$

$$\rho = \frac{M_1^2}{4(\chi_2 M_2 - 2M_0)} + \frac{c_0}{a^3} \left[ \frac{|M_1|}{\chi_2 M_2 - 2M_0} \right]^{1/2} + O\left( \frac{c_0^2}{a^6} \right), \quad (51)$$

$$p = -\frac{M_1^2}{4(\chi_2 M_2 - 2M_0)} + O\left( \frac{c_0^2}{a^6} \right), \quad (52)$$
and from the Friedmann eqs. (46) (here we need $M_1 = -|M_1|$):

$$a(t) \sim \left( \frac{\tilde{C}_0}{2\Lambda} \right)^{1/3} \sinh^{2/3} \left( \sqrt{\frac{3}{4} \Lambda} t \right), \quad \dot{\varphi} \sim \text{const} \sinh^{-2} \left( \sqrt{\frac{3}{4} \Lambda} t \right),$$

effective CC $\Lambda \equiv \frac{M_1^2}{8(\chi_2 M_2 - 2M_0)}$, $\tilde{C}_0 \equiv c_0 \left[ \frac{|M_1|}{\chi_2 M_2 - 2M_0} \right]^{1/2}$.

Now we will extend our results from the previous two sections by considering a combination of the both models above (14) and (1) – gravity coupled to both “inflaton” and “darkon” scalar fields within the non-Riemannian volume-form formalism, as well as we will also add coupling to the bosonic sector of the electro-weak model:

$$S = \int d^4 x \Phi(A) \left[ g^{\mu\nu} R_{\mu\nu}(\Gamma) + L_1(\varphi, X) - g^{\mu\nu} (\nabla_\mu \sigma_a)^* \nabla_\nu \sigma_a - V_0(\sigma) \right] +$$

$$\int d^4 x \Phi(B) \left[ L_2(\varphi, X) - \frac{1}{4g^2} F^2(A) - \frac{1}{4g'^2} F^2(B) + \frac{\Phi(H)}{\sqrt{-g}} \right]$$

$$+ \int d^4 x (\sqrt{-g} + \Phi(C)) L(u, Y).$$  (31)
\( \sigma \equiv (\sigma_a) \) is a complex \( SU(2) \times U(1) \) iso-doublet scalar field with the isospinor index \( a = +, 0 \) indicating the corresponding \( U(1) \) charge. The gauge-covariant derivative acting on \( \sigma \) reads:

\[
\nabla_\mu \sigma = \left( \partial_\mu - \frac{i}{2} \tau_A A^A_\mu - \frac{i}{2} B_\mu \right) \sigma ,
\]

with \( \frac{1}{2} \tau_A \) (\( \tau_A \) - Pauli matrices, \( A = 1, 2, 3 \)) indicating the \( SU(2) \) generators and \( A^A_\mu \) (\( A = 1, 2, 3 \)) and \( B_\mu \) denoting the corresponding \( SU(2) \) and \( U(1) \) gauge fields.

- The "bare" \( \sigma \)-field potential is of the same form as the standard Higgs potential:

\[
V_0(\sigma) = \frac{\lambda}{4} ((\sigma_a)^* \sigma_a - \mu^2)^2 .
\]

- The gauge field kinetic terms are (all indices \( A, B, C = (1, 2, 3) \)):

\[
F^2(A) \equiv F^A_{\mu\nu}(A) F^A_{\kappa\lambda}(A) g^{\mu\kappa} g^{\nu\lambda} , \quad F^2(B) \equiv F_{\mu\nu}(B) F_{\kappa\lambda}(B) g^{\mu\kappa} g^{\nu\lambda} ,
\]

\[
F^A_{\mu\nu}(A) = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu + \epsilon^{ABC} A^B_\mu A^C_\nu , \quad F_{\mu\nu}(B) = \partial_\mu B_\nu - \partial_\nu B_\mu .
\]
Notice that in the choice of the potential 33 we have for first time introduced explicit spontaneous breaking of scale invariance. Otherwise we get a vacuum manifold problem, as a consequence that the fact that the effective potential is a joint potential for all fields and is a perfect square, so whatever it is a square of defines a vacuum manifold. The vacuum manifold problem can be avoided, while respecting scale invariance by a linear coupling of dilaton to the Gauss Bonnet density. E.G. , S. Rajpoot and H.Nishino, to appear in E.J. P. C.
In this case the dilaton gets an additional potential, which does not contribute to the vacuum energy density, only breaks the vacuum manifold degeneracy but does not spoil the nice features concerning of the theory concerning the CCP. Similar effect can be achieved by assigning the dilaton a coupling to a topological gauge density, this coupling does not involve the metric, and as in axions it gives the dilaton an additional potential, but still the CC is determined by the old potential at a somewhat shifted value of dilaton field, E.G., S. Rajpoot and H. Nishino, PLB.
Following the same steps as above, we derive from (31) the physical Einstein-frame theory w.r.t. Weyl-rescaled Einstein-frame metric $\bar{g}_{\mu\nu}$ (19) and perform an additional “darkon” field redefinition $u \rightarrow \tilde{u}$:

$$\frac{\partial \tilde{u}}{\partial u} = (V(u) - 2M_0)^{-\frac{1}{2}} ; \quad Y \rightarrow \tilde{Y} = -\frac{1}{2} \bar{g}^{\mu\nu} \partial_\mu \tilde{u} \partial_\nu \tilde{u} . \quad (35)$$

The Einstein-frame action reads:

$$S = \int d^4x \sqrt{-\bar{g}} \left[ R(\bar{g}) + L_{\text{eff}}(\varphi, \bar{X}, \bar{Y}; \sigma, A, B) \right] , \quad (36)$$

where (recall $\bar{X} = -\frac{1}{2} \bar{g}^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi$):

$$L_{\text{eff}}(\varphi, \bar{X}, \bar{Y}; \sigma, A, B) = \bar{X} - \bar{Y} \left( V_1(\varphi) + V_0(\sigma) + M_1 - \chi_2 be^{-\alpha \varphi} \bar{X} \right)$$
$$+ \bar{Y}^2 \left[ \chi_2 (U(\varphi) + M_2) - 2M_0 \right] + L[\sigma, A, B] , \quad (37)$$

with:

$$L[\sigma, A, B] \equiv -\bar{g}^{\mu\nu}(\nabla_\mu \sigma_a)^* \nabla_\nu \sigma_a - \frac{\chi_2^2}{4g^2} F^2_\sigma (A) - \frac{\chi_2^2}{4g^{r^2}} F^2_\sigma (B) . \quad (38)$$

The Lagrangian (37) is again of a generalized “k-essence” form (non-linear w.r.t. both “inflaton” and “darkon” kinetic terms $\bar{X}$ and $\bar{Y}$). $M_0$ and $M_1, M_2, \chi_2$ are the same integration constants as in (4) and (21), respectively.

The action (36)-(37) possesses an obvious Noether symmetry under the shift $\tilde{u} \rightarrow \tilde{u} + \text{const}$ with current conservation:
which is Einstein-frame counterpart of the original $g_{\mu\nu}$-frame “dust” dark matter density conservation (7).

For static (spacetime independent) scalar field configurations (here the original “darkon” field $u$ is static, whereas the transformed one $\tilde{u}$ (35) is not – this is due to the dynamical Lagrangian “darkon” constraint (4)) we have:

$$\bar{Y}_{\text{static}} = \frac{V_1(\varphi) + V_0(\sigma) + M_1}{2\chi_2(U(\varphi) + M_2) - 4M_0},$$

which upon substitution into (37) yields the following total scalar field effective potential (cf. Eq.(25)):

$$U_{\text{eff}}(\varphi, \sigma) = \frac{\left( V_1(\varphi) + V_0(\sigma) + M_1 \right)^2}{4[\chi_2(U(\varphi) + M_2) - 2M_0]}$$

As for the purely “inflaton” potential (25), the “inflaton+Higgs” potential (41) similarly possess two infinitely large regions: (−) flat region for large negative and (+) flat region and large positive values of the “inflaton”, respectively, as in Fig.1 (when $\sigma$ is fixed).
In the $(+)$ flat region (41) reduces to (cf. (27)):

$$U_{\text{eff}}(\varphi, \sigma) \simeq U_{(+)}(\sigma) = \frac{(\frac{\Lambda}{4} ((\sigma_a)^* \sigma_a - \mu^2)^2 + M_1)^2}{4(\chi_2M_2 - 2M_0)},$$

(42)

which obviously yields as a lowest lying vacuum the Higgs one:

$$|\sigma| = \mu,$$

(43)

i.e., in the “late” (post-inflationary) Universe we have the standard spontaneous breakdown of $SU(2) \times U(1)$ gauge symmetry. Moreover, at the Higgs vacuum (43) we obtain from (42) a dynamically generated cosmological constant $\Lambda_{(+)}$ of the “late” Universe:

$$U_{(+)}(\mu) \equiv 2\Lambda_{(+)} = \frac{M_1^2}{4(\chi_2M_2 - 2M_0)}.$$

(44)

In the $(-)$ flat region (41) reduces to the same expression as in (26), which is $\sigma$-field independent. Thus, the Higgs-like iso-doublet scalar field $\sigma_a$ remains massless in the “early” (inflationary) Universe and accordingly does not contribute to the quark potential energy at all.
5th FORCE PROBLEM

We have shown that dilaton couplings do not introduce a 5th FORCE PROBLEM, that is a long range interaction, even if no mass is generated. The effective coupling automatically vanishes for matter denser than the vacuum energy density.

E.G. and A. Kaganovich
References. SSB of scale invariance in Two Measures Theory

_Scale invariance, new inflation and decaying lambda terms_
DOI: [10.1142/S0217732399001103](https://doi.org/10.1142/S0217732399001103)
e-Print: [gr-qc/9901017](https://arxiv.org/abs/gr-qc/9901017) | [PDF](https://www.worldscientific.com/doi/pdf/10.1142/S0217732399001103)
Non Singular Cosmology, Unified Picture of Inflation and Dark Energy


*arXiv:1408.5344* [gr-qc]

EG, Ramon Herrera, Pedro Labrana, Emil Nissimov and Svetlana Pacheva,
Two Measures Models of Dark Matter/
Dark Energy

A Two Measure Model of Dark Energy and Dark Matter. E. Guendelman, D. Singleton and N. Yongram, Published in JCAP 1211 (2012) 044
DOI: 10.1088/1475-7516/2012/11/044
e-Print: arXiv:1205.1056 [gr-qc]
Relating Dark Matter to a Noether Symmetry

DOI: 10.1140/epjc/s10052-015-3699-8
e-Print: arXiv:1508.02008 [gr-qc]
Including all effects, Inflation DE unification, DE/DM and Higgs effect

**Quintessential Inflation, Unified Dark Energy and Dark Matter, and Higgs Mechanism**


e-Print: [arXiv:1609.06915](https://arxiv.org/abs/1609.06915) [gr-qc]
Generalize TMT to include possibility of interactive DE/DM scenarios

Interacting Diffusive Unified Dark Energy and Dark Matter from Scalar Fields
David Benisty, E.I. Guendelman

e-Print: arXiv:1701.08667 [gr-qc]

May be another talk on this..
also look at a previous paper and references in both papers and in

Unification of Inflation and Dark Energy from Spontaneous Breaking of Scale Invariance

Eduardo Guendelman, Emil Nissimov, Svetlana Pacheva, Jul 23, 2014

Recall on wider applications of alternative measures

- (i) Study of $D = 4$-dimensional models of gravity and matter fields containing the new measure of integration (1), which appears to be promising candidates for resolution of the dark energy and dark matter problems, the fifth force problem, *etc.*

- (ii) Study of a new type of string and brane models based on employing of a modified world-sheet/world-volume integration measure. It allows for the appearance of new types of objects and effects like, for example, a spontaneously induced variable string tension.

- (iii) Studying modified supergravity models. Here we will find some outstanding new features: (a) the cosmological constant arises as an arbitrary integration constant, totally unrelated to the original parameters of the action, and (b) spontaneously breaking of local supersymmetry invariance.
THANK YOU FOR YOUR ATTENTION !!

NEXT, SECOND TALK
There have been theoretical approaches to gravity theories, where a fundamental constraint is implemented. For example in the Two Measures Theories [1]-[9] one works, in addition to the regular measure of integration in the action $\sqrt{-g}$, also with another measure which is also a density and which is also a total derivative. In this case, one can use for constructing this measure 4 scalar fields $\varphi_a$, where $a = 1, 2, 3, 4$. Then we can define the density $\Phi = \varepsilon^{\alpha\beta\gamma\delta}\varepsilon_{abcd}\partial_\alpha\varphi_a\partial_\beta\varphi_b\partial_\gamma\varphi_c\partial_\delta\varphi_d$, and then we can write an action that uses both of these densities:

$$S = \int d^4x \Phi \mathcal{L}_1 + \int d^4x \sqrt{-g} \mathcal{L}_2$$

(1)

As a consequence of the variation with respect to the scalar fields $\varphi_a$, assuming that $\mathcal{L}_1$ and $\mathcal{L}_2$ are independent of the scalar fields $\varphi_a$, we obtain that:

$$A_a^\alpha \partial_\alpha \mathcal{L}_1 = 0$$

(2)

where $A_a^\alpha = \varepsilon^{\alpha\beta\gamma\delta}\varepsilon_{abcd}\partial_\beta\varphi_b\partial_\gamma\varphi_c\partial_\delta\varphi_d$. Since $\text{det}[A_a^\alpha] \sim \Phi^3$ as one easily see, then that for $\Phi \neq 0$, (2) implies that $\mathcal{L}_1 = M = \text{Const}$. This result can expressed as a covariant conservation of a stress.
energy momentum of the form $T_{(\Phi)}^{\mu \nu} = \mathcal{L}_1 g^{\mu \nu}$, and using the 2nd order formalism, where the covariant derivative of $g_{\mu \nu}$ is zero, we obtain that $\nabla_\mu T_{(\Phi)}^{\mu \nu} = 0$, implies $\partial_\alpha \mathcal{L}_1 = 0$. This suggests generalizing the idea of the Two Measures Theory, by imposing the covariant conservation of a more nontrivial kind of energy momentum tensor, which we denote as $T_{(\chi)}^{\mu \nu}$ [11]. Therefore we consider an action of the form:

$$S = S_{(\chi)} + S_{(R)} = \int d^4x \sqrt{-g} \chi_{\mu,\nu} T_{(\chi)}^{\mu \nu} + \frac{1}{16\pi G} \int d^4x \sqrt{-g} R$$

(3)

where $\chi_{\mu,\nu} = \partial_\nu \chi_{\mu} - \Gamma_{\mu \nu}^\lambda \chi_\lambda$. If we assume $T_{(\chi)}^{\mu \nu}$ to be independent of $\chi_\mu$ and having $\Gamma_{\mu \nu}^\lambda$ being defined as the Christoffel Connection Coefficients, then the variation with respect to $\chi_\mu$ gives a covariant conservation $\nabla_\mu T_{(\chi)}^{\mu \nu} = 0$. Notice the fact that the energy density is the canonically conjugated variable to $\chi^0$, which is what we expect from a dynamical time (here represented by the dynamical time $\chi^0$). Some cosmological solutions of (3) have been studied in [12], in the context of specially flat radiation like solutions, and considering a gauge field equations in curved space time.

For a related approach where a set of dynamical space-time coordinates are introduced, not only in the measure of integration, but also in the lagrangian, see [13].

It is of interest to introduce also a mechanism that produces non conserved energy momentum tensors which can lead eventually to a formulation of interacting DE-DM models. To start we discuss a toy model in one dimension describing a system that allows the non conservation of a certain energy functional, which increases or decreases linearly with time, while there is another energy which is conserved.
II. A MECHANICAL SYSTEM WITH A CONSTANT POWER AND DIFFUSIVE PROPERTIES

In order to see the applications of the ideas, we start with a simple action of one dimensional particle in a potential $V(x)$. We introduce a coupling between the total energy of the particle $\frac{1}{2}m\dot{x}^2 + V(x)$ and the second derivative of some dynamical variable $B$:

$$S = \int \dot{B}[\frac{1}{2}m\dot{x}^2 + V(x)]dt$$

(4)

The equation of motion according to the dynamical variable $B$, give that the second derivative of the total energy is zero. In other words, the total energy of the particle is linear in time:

$$\frac{1}{2}m\dot{x}^2 + V(x) = E(t) = Pt + E_0$$

(5)

where $P$ is a constant power which given to the particle or taken from it, and $E_0$ is the total energy of the particle at time equals zero.

From the equation of motion according to coordinate $x$ we get a close connection between the dynamical variable $B$ and the coordinate of the particle:

$$m\ddot{x}\frac{d^2B}{dt^2} + m\dot{x}\frac{d^3B}{dt^3} = V'(x)\frac{d^2B}{dt^2}$$

(6)

with the equation (5) give:

$$\frac{\dot{B}}{\dot{B}} = \frac{2V'(x)}{\sqrt{2m(E(t) - V(x))}} - \frac{P}{2(E(t) - V(x))}$$

(7)
To get a feeling of these kind of theories, let us look at the case of a harmonic oscillator \( V(x) = \frac{1}{2}kx^2 \). First of all, we see from eq (5) and the condition that the right hand side be positive, since the left hand side obviously is positive, we get that there is a boundary time \( \tau = -\frac{E_0}{P} \), that for \( P > 0 \) we get \( t > \tau \), and for \( P < 0 \) there is a maximal time \( t < \tau \). Let us consider the case the power \( P \) is positive. The amplitude, which is the maximum value of the coordinate \( x \), behaves as square-root of the total energy \( A = \sqrt{\frac{2E}{m}} \). For large time \( t \gg \tau \), the energy absorbed by the system in each oscillation is much smaller than the total energy of the system \( PT \ll E(t) \). In this case, the total energy is proportional to time, and the amplitude is \( A \propto \sqrt{t} \). Since the time has to be proportional to the number of oscillations times the period of each oscillation, then \( N \), and since the period of the oscillations does not depend on the energy \( A \propto \sqrt{N} \). This is exactly the behavior of Brownian motion \( \bar{x}^2 \sim N \), which is the basis of the well known diffusion theory. As in the standard diffusion theory, we started with a particle at \( x = 0 \) which corresponds to the time \( E(t) = 0 \). As the system acquires energy it diffuses to larger values of \( x \).

Following this notion, at the classical level, following the approach in [14][15], we can calculate the probability of this harmonic oscillator being between \( x \) and \( x + dx \). The probability density is equal to the ratio between the time the particle staying at that interval, and the oscillation period \( T = \frac{2\pi}{\omega} \). The total energy is given as equation (5), and therefore the velocity of the particle is \( \dot{x} = \sqrt{\frac{2P t}{m} - \omega^2 x^2} \), and the probability is \( \rho(x)dx = \frac{2dx}{|\rho|T} = \left[\pi(A^2 - x^2)\right]^{-1} dx \) for \( |x| < A \), and \( \rho(x) = 0 \) otherwise. Then the average of \( x^2 \) after integration will become:

\[
\langle x^2 \rangle = \int_{-A}^{+A} \rho(x)x^2dx = 2Dt
\]
when $D = \frac{p}{m}$. This is precisely as in Brownian motion. The momentums for this toy model are:

$$\pi_x = \frac{\partial L}{\partial \dot{x}} = m \dot{x} \dot{B}$$  \hspace{1cm} (9)$$

$$\pi_B = \frac{\partial L}{\partial \dot{B}} - \frac{d}{dt} \frac{\partial L}{\partial B} = -\frac{d}{dt} E(t)$$  \hspace{1cm} (10)$$

$$\Pi_B = \frac{\partial L}{\partial B} = E(t)$$  \hspace{1cm} (11)$$

Using Hamiltonian formalism (with second order derivative [16][17]) we get that the hamiltonian of the system is:

$$\mathcal{H} = \dot{x}\pi_x + \dot{B}\pi_B + \Pi_B - L = \pi_x \sqrt{\frac{2}{m}(\Pi_B - V(x))} + \dot{B}\Pi_B = \text{Const}$$  \hspace{1cm} (12)$$

Since the action in not dependent explicitly on time, the hamiltonian is conserved. So even that the total energy of the particle is not conserved, there is the conserved hamiltonian (12). Notice that this hamiltonian is not necessarily bounded from below. But the total energy $E(t)$, which is not grows or decreases linearly, but is always positive if the potential is positive and the growth of the pertubations is according to a Brownian type behavior, not an unstable exponentially growing behavior. A generalization of this notion to generally coordinate invariant models will give us similar phenomena. In this case, the hamiltonian is automatically zero, as a consequence of re-parametrization invariance.
Let's consider a 4 dimensional case, where there is a coupling between a scalar field $\chi$, and a stress energy momentum tensor $T_{(\chi)}^{\mu \nu}$:

$$S_{(\chi)} = \int d^4x \sqrt{-g} \chi,_{\mu}^{\nu} T_{(\chi)}^{\mu \nu}$$  \hspace{1cm} (13)$$

where $\chi,_{\mu}^{\nu}$ are covariant derivative of the scalar field. When $\Gamma_{\mu \nu}^{\lambda}$ is being defined as the Christoffel Connection Coefficients, the variation with respect to $\chi$ gives a covariant conservation of a current $f^\mu$:

$$\nabla_\mu T_{(\chi)}^{\mu \nu} = f^\nu; \nabla_\nu f^\nu = 0$$  \hspace{1cm} (14)$$

which it is the source of the stress energy momentum tensor. This corresponds to the "dynamical space time" theory (3), where the dynamical space time 4-vector $\chi_\mu$ is replaced by a gradient of a scalar field $\chi$. In the "dynamical space theory" we obtain 4 equations of motion, by the variation of $\chi_\mu$, which correspond to covariant conservation of energy momentum tensor $\nabla_\mu T_{(\chi)}^{\mu \nu} = 0$. By changing the 4 vector to a gradient of a scalar $\partial_\mu \chi$ at the end, what we do is to change the conservation of energy momentum tensor to asymptotic conservation of energy momentum tensor (14) which corresponds to a conservation of a current $\nabla_\nu f^\nu = 0$. In an expanding universe, the current $f_\mu$ gets diluted, so then we recover asymptotically a covariant conservation law for $T_{(\chi)}^{\mu \nu}$ again.

These equations (14) have close correspondence with those obtained in a "diffusion scenario" for DE-DM exchange [19][20]. Other models of DE-EM interactions have been studied in [21][22]. Those approaches are not based on an action principle.
This stress energy tensor is substantially different from stress energy tensor we all know, which is defined as \( T_{(G)}^{\mu \nu} = R^{\mu \nu} - \frac{1}{2} g^{\mu \nu} R \). Like the mechanical system, where the total energy of the particle is not conserved (but there is some constant power to the system) and there is some hamiltonian which it does keep conservation, in the 4-D case, the stress energy momentum tensor \( T_{(\chi)}^{\mu \nu} \) is not conserved (but there is some conserved current \( f^\mu \), which is the source to this stress energy momentum tensor non conservation), here there is some conserved stress energy tensor \( T_{(G)}^{\mu \nu} \) which comes from variation of the action according to the metric:

\[
T_{(G)}^{\mu \nu} = \frac{-2}{\sqrt{-g}} \frac{\delta \left( \sqrt{-g} \mathcal{L}_M \right)}{\delta g^{\mu \nu}}; \nabla_\mu T_{(G)}^{\mu \nu} = 0
\]  

(15)

The lagrangian \( \mathcal{L}_M \) could be the modified term \( \chi_{, \mu ; \nu} T_{(\chi)}^{\mu \nu} \), but as we will see, it could be added to more action terms. Using different expressions for \( T_{(\chi)}^{\mu \nu} \) which depend on another variables, will give the connection between the scalar filed \( \chi \) and those other variables.

Notice that for the theory the shift symmetry holds, and

\[
\chi \rightarrow \chi + C_\chi; \quad T_{(\chi)}^{\mu \nu} \rightarrow T_{(\chi)}^{\mu \nu} + g^{\mu \nu} C_T
\]  

(16)

will not change any equation of motion. when \( C_\chi, C_T \) are some arbitrary constants. This means that if the matter is coupled through its energy momentum tensor as in (13), a process of redefinition of the energy momentum tensor, will not affect the equations of motion. Of course such type of redefinition of the energy momentum tensor is exactly what is done in the process of normal ordering in Quantum Field Theory for example.
Our starting point is the following non-conventional gravity-scalar-field action, which will produce a diffusive type of interacting DE-DM theory:

\[ S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R + \int \sqrt{-g} \mathcal{L}(\phi, X) + \int d^4x \sqrt{-g} \chi_{\mu\nu} T_{(\chi)}^{\mu\nu} \]  

(17)

with the following explanations for the different terms: \( R \) is the Ricci scalar, and give us Einstein-Hilbert action. \( \mathcal{L}(\phi, X) \) is general-coordinate invariant Lagrangian of a single scalar field \( \phi \), which can be of an arbitrary generic "k-essence" type: some function of a scalar filed \( \phi \) and the combination \( X = \partial_\mu \phi \partial^{\mu} \phi \) [23][24][25]):

\[ \mathcal{L}(\phi, X) = \sum_{N=1}^{\infty} A_n(\phi) X^n - V(\phi) \]  

(18)

As we will see, this last action will produce a diffusive interaction between DE-DM type theory. For the anzats of \( T_{(\chi)}^{\mu\nu} \) we chose using some tensor proportional to the metric, with a proportionality function \( \Lambda(\phi, X) \):

\[ T_{(\chi)}^{\mu\nu} = g^{\mu\nu} \Lambda(\phi, X) \Rightarrow S_{(\chi)} = \int d^4x \Lambda \sqcup \chi \]  

(19)

From the variation of the scalar field \( \chi \) we get: \( \sqcup \Lambda = 0 \), whose solution will be interpreted as a dynamically generated Cosmological Constant with diffusive source.

We take the simple example for this generalized thoery, and for the functions \( \mathcal{L}, \Lambda \) we take the first order of the Taylor expansion from (18), or \( \mathcal{L} = \Lambda = X (A_1 = 1) \). From the variation according to the scalar field we get a conserved current \( j_{\mu}^\alpha = 0 \):

\[ j_\alpha = 2(\sqcup \chi + 1) \phi,\alpha \]  

(20)
For a cosmological solution we take into account only change as function of time \( \phi = \phi(t) \). From that we get that the '0' component of the current \( j_\alpha \) is non zero. The last variation we should take is according to the metric (using the identities at appendix A), which gives us a conserved stress energy tensor:

\[
T^{\mu\nu}_{(G)} = g^{\mu\nu}(-\Lambda + \chi^{\sigma\lambda} \Lambda_{,\sigma}) + j^{\mu} \phi^{\nu} - \chi^{\nu\mu} \Lambda^{\nu} - \chi^{\nu\lambda} \Lambda^{\mu}
\]

For a cosmological solutions the interpretation for dark energy is for term proportional to the metric \(-\Lambda + \chi^{\sigma\lambda} \Lambda_{,\sigma}\), and dark matter dust from the '00' component of the tensor \(j^{\mu}\phi^{\nu} - \chi^{\nu\mu} \Lambda^{\nu} - \chi^{\nu\lambda} \Lambda^{\mu}\). Lets see this solution for Fridman Robertson Walker Metric:

\[
ds^{2} = -dt^{2} + a^{2}(t)[\frac{dr^{2}}{1-kr^{2}} + r^{2}d\Omega^{2}]
\]

The basic combination becomes \( \mathcal{L} = \Lambda = X = \partial_{\mu} \phi \partial^{\mu} \phi = -\dot{\phi}^{2} \). We get that the variation of the scalar field (19) will give:

\[
2\dot{\phi} \ddot{\phi} = \frac{C_{2}}{a^{3}}
\]

which can be integrated:

\[
\phi^{2} = C_{1} + C_{2} \int \frac{dt}{a^{3}}
\]

The conserved current from eq (20) will give us the relation:

\[
2\dot{\phi}(\nabla \chi + 1) = \frac{C_{3}}{a^{3}}
\]
which can be also integrated to give:

\[ \dot{\chi} = \frac{C_4}{a^3} + \frac{1}{a^3} \int a^3 dt - \frac{C_3}{2a^3} \int \frac{dt}{\phi} \]  

which gives us the anzats for the scalar field \( \chi \). From eq (20) we get the terms for DE-DM densities:

\[ \rho_{de} = \phi^2 - 2\dot{\chi}\phi\ddot{\phi}, p_{de} = -\rho_{de} \]  

\[ \rho_{dm} = \frac{C_3}{a^3} \phi + 4\dot{\chi}\phi\ddot{\phi}, p_{dm} = 0 \]

This lead to a Fridman equations with (27)(28) as source, and there a few solutions that be we want to discuss about.

**V. Perturbative solution**

One simple case is when \( C_2 = 0 \). That means that the dark energy of this universe is constant \( \phi^2 = C_1 \) and \( \phi = 0 \). The equation of motions for the dark energy and dust (27-28) are independent on the scalar filed \( \chi \), and therefor the density of dust is that universe is \( \frac{C_3\sqrt{C_1}}{a^3} \). This solution says there is no interaction between dark energy and dark matter. This is precisely the solution of Two Measure Theory [26][27][28], with the action:

\[ S = \int \sqrt{-g} R + \int (\Phi + \sqrt{-g})\mathcal{L}(X,\phi) \]  

which provides a unified picture of DE-DM. More about Two Measure Theory and related models and solutions for DE-DM see a discussion in appendix C. The FRWM for both theories gives the solution (general case for TMT, but a special solution for the Diffusive Interaction Theory, where we set the diffusion parameter \( C_2 \) to zero):
\[ \rho_{DE} = \phi^2 = C_1 \]  
\[ \rho_{Dust} = \frac{\sqrt{C_1 C_3}}{a^3} \]  

(30)  
(31)

The conclusion from this correspondence is that the diffusion between dark energy and dark matter dust at the late universe is very small, since that is the effect of the \( C_2 \) term, and therefore we can estimate the solution by perturbation theory. The exact solution of the case of standard dark energy and dust, using (30)(31) is [29]:

\[ a_0(t) = \left( \frac{C_3}{\sqrt{C_1}} \right)^{\frac{1}{3}} \sinh^{3/2}(\alpha t) \]  
\[ \alpha = \frac{3}{2} \sqrt{C_1} \]  

(32)

where \( \alpha = \frac{3}{2} \sqrt{C_1} \) and the perturbations for the dark energy and the correction of the normalization of the dust, inserting the zeroth order solution (30)(31) into the general equations with \( C_2 \) different from zero (23-28). We obtain that the difference from the original DE and DM (30)(31) densities are:

\[ \Delta \Lambda = -C_2 \left[ \frac{C_4}{\sinh^4(\alpha t)} + \frac{\sinh(\alpha t) - 2\alpha t}{4\alpha \sinh^4(\alpha t)} + \frac{C_3}{2\sqrt{C_1} \sinh^4(\alpha t)} + \frac{1}{\alpha \coth(\alpha t)} \right] \]  
\[ \Delta \Lambda = \frac{2C_3 C_2}{3C_1} \coth(\alpha t) \]  

(33)  
(34)
We can see from those terms, that for a deviation from the unperturbed standart solution, the behavior of dark enery and dust are opposite - for rising dark energy (for example the components are $C_2 < 0$; $C_1, C_3, C_4 > 0$), the dark matter amount $(a^3 \rho_{dm})$ goes lower. Or in case of decreasing dark energy, the amounts of dark matter goes up (and $C_1, C_2, C_3, C_4 > 0$).

VI. DIFFUSIVE DARK ENERGY AND DUST

A related approach, but without using the action principle, is a diffusive dark energy and dust picture which was formulated by Calogero [19][20]. The claim is that the total energy momentum tensor which appers in Einstein equation (in our terminology $T^{\mu \nu}_{(G)}$) is conserved. But for the dark energy and dust stress tensors there is some current which is the source of those tensors (together they conserved):

$$-\nabla_\mu T^{\mu \nu}_{(\Lambda)} = \nabla_\mu T^{\mu \nu}_{(Dust)} = J^\nu, J^\nu_{;\nu} = 0$$ (35)

The solution for this equation gives the folowing dependence between the density and the scale parameter:

$$\rho_{de} = C_1 + C_2 \int \frac{dt}{a^3}$$ (36)

$$\rho_{dm} = \frac{C_3}{a^3} - \frac{C_2 t}{a^3}$$ (37)

A complete set of solutions of these differential equations (in the form of Fridman equations) is very complicated, but one phenomenological solution for this theory predicts a DE-DM similar ratio to the observed one [30][31].
Both approaches (which are described in this paper and in Calogero's theory) become very similar when the time derivative of the scalar field is very low $\dot{\chi} \ll 1$ or the scale parameter is very large. In that case

$$\rho_{de} = C_1 + C_2 \int \frac{dt}{a^3}$$

The dark matter dust will reduce to the term:

$$\rho_{dm} = \frac{C_3}{a^3} \phi$$

and for those equation implies a diffusion between dark energy and dark matter dust, like Calogero has found. In this model they assumed that the dark enegy and the dust are not separatly conserved.

As we can see, the term for dark energy has a precise correspondance in our model and Calogero-model. The term for dust is quite similar only when the integration $\int \frac{dt}{a^3}$ will be well approximated by a simple product $\frac{t}{a^3}$. By solving this condition, we get that it is a good pertubative solution only when:

$$\frac{\dot{a}}{a} (t - t_0) \ll 1$$

VII. Discussion, Conclusions and Prospects

In this paper we have generalized the TMT and the dynamical space time theory, which imposes the covariant conservation of an energy momentum tensor. By demanding that dynamical space time 4-vector $\chi_\mu$, that appears in the dynamical space time theory be a gradient $\partial_\mu \chi$. We don't obtain the covariant conservation of energy momentum tensor that is introduced in the action.
Instead we obtain a current conservation. The current being the divergence of this energy momentum tensor. This current that drives the non-conservation of the energy momentum tensor, is dissipated in the case of an expanding universe. So we get an asymptotic conservation of this energy momentum tensor. Because the four divergence of the covariant divergence of both the dark matter and dark energy is zero, we can make contact with the dissipative models of [19][20]. This can give deeper motivation for these models motivation for these kind of models and allow the construction of new models.

This energy tensor, in not the gravitational energy tensor which appears in the right hand side of the Einstein tensor, in the gravity equations. But the non covariant conservation of the energy momentum tensor that appears in the action induces an energy momentum transfer between the dark energy and dark matter components, of the gravitational energy momentum tensor. In a way that resembles the ideas in [30][31]. But they don’t provide any action principle to support their ideas. Although the mechanism is similar, our formulation and theirs are not equivalent.

We have seen that asymptotically the behavior of dark energy and dust are different - for rising dark energy (for example the components are $C_2 < 0; C_1, C_3, C_4 > 0$), the dark matter amount $(a^3 \rho_{dm})$ goes lower. Or in case of decreasing dark energy, the amounts of dark matter goes up (and all the constants of integration are positive).

In the future we will study not only the asymptotic behavior, but the full numerical solution of the dark energy and dark matter components, starting from the early universe
Interacting Diffusive Unified Dark Energy and Dark Matter from Scalar Fields

David Benisty and E.I. Guendelman
benidav@post.bgu.ac.il and guendel@post.bgu.ac.il
Department of Physics, Ben Gurion University of the Negev
Beer-Sheva 84105, Israel

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