Kinky Galileons

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I discuss classical traveling wave solutions of hypothetical scalar field "galileon" models, in flat space-time. I construct a hierarchy of models with potentials that admit such solutions.

Based on work done in collaboration with David Fairlie, University of Durham.

Je n'ai fait celle-ci plus longue que parce que je n'ai pas eu le loisir de la faire plus courte.
B Pascal, *Lettres Provinciales XVI* (1656)

Simplest example

Recall from my BASIC 2016 talk, Galileon theories are a class of models for hypothetical scalar fields whose Lagrangians involve multilinears of first *and* second derivatives, but whose *nonlinear* field equations are still *only second order*. They may be important for the description of large-scale features in astrophysics as well as for building models in elementary particle theory.

The simplest example involves a single field.

$$\mathcal{A}_2 = \int \phi_\alpha \phi_\alpha \; \phi_{\beta\beta} \; d^n x \tag{1}$$

where ϕ is the scalar galileon field, $\phi_\alpha = \partial\phi\left(x\right)/\partial x^\alpha$, etc., and where repeated indices are summed using the Lorentz metric $\delta_{\mu\nu} = \operatorname{diag}\left(1, -1, -1, \cdots\right)$. Note that $\mathcal{A}_2$ is invariant, modulo boundary effects, under $\delta\phi = c + u_\gamma x_\gamma$, where u_γ is a constant vector. Hence the name "galileon" field.

The field equations are

$$0 = \mathcal{E}_2\left[\phi\right] \equiv \phi_{\alpha\alpha}\phi_{\beta\beta} - \phi_{\alpha\beta}\phi_{\alpha\beta} \tag{2}$$

So, obviously, $\mathcal{E}_2\left[c + u_\gamma x_\gamma\right] = 0$ again where u_γ is a constant vector. More generally,

$$\mathcal{E}_2\left[f\left(c + u_\gamma x_\gamma\right)\right] = 0 \tag{3}$$

for any twice differentiable function f.

Generalizations

There is a hierarchy for $1 \leq k \leq n$.

$$\mathcal{A}_k = \int \phi_\alpha \phi_\alpha \ \mathcal{E}_{k-1} [\phi] \ d^n x \tag{4}$$

where $\mathcal{E}_0 [\phi] = 1$, $\mathcal{E}_1 [\phi] = \phi_{\alpha\alpha}$, $\mathcal{E}_2 [\phi]$ is as above, and for other k only 2nd derivatives appear as

$$\mathcal{E}_k [\phi] = \delta^{\alpha_1 \alpha_2 \cdots \alpha_k}_{\beta_1 \beta_2 \cdots \beta_k} \times \phi_{\alpha_1 \beta_1} \phi_{\alpha_2 \beta_2} \cdots \phi_{\alpha_k \beta_k} \tag{5}$$

where $\delta^{\alpha_1 \alpha_2 \cdots}_{\beta_1 \beta_2 \cdots}$ is a generalized Kronecker symbol. All $\mathcal{A}_k$ are invariant under "galileon" variations, and have the same elementary solutions

$$\mathcal{E}_k [f (c + u_\mu x_\mu)] = 0 \tag{6}$$

All these field equations follow from expanding a determinant,

$$\det (\mathbf{1} + \lambda \partial\partial\phi) = \sum_{k=0}^{n} \frac{\lambda^k}{k!} \ \mathcal{E}_k (\partial\partial\phi) \tag{7}$$

with $\mathcal{E}_0 \equiv 1$, where by "$\partial\partial\phi$" I mean the $n \times n$ Hessian matrix of second partial derivatives in n dimensions.

In fact, for any matrix M

$$\det\left(\mathbf{1}+\lambda M\right)=\sum_{k=0}^{n}\frac{\lambda^{k}}{k!}\ \mathcal{E}_{k}\left(M\right)\ \ , \tag{8}$$

the coefficients $\mathcal{E}_k\left(M\right)$ for all $k\leq n$ can be expressed in terms of *another* set of determinants, for an auxiliary $k\times k$ matrix containing the various traces, $\mathcal{T}_m\equiv\mathrm{Tr}\left(M^m\right)$:

$$\mathcal{E}_{k}\left(M\right)=\det\begin{pmatrix} \mathcal{T}_1 & k-1 & 0 & \cdots & 0 & 0 & 0 \\ \mathcal{T}_2 & \mathcal{T}_1 & k-2 & \cdots & 0 & 0 & 0 \\ \mathcal{T}_3 & \mathcal{T}_2 & \mathcal{T}_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathcal{T}_{k-2} & \mathcal{T}_{k-3} & \mathcal{T}_{k-4} & \cdots & \mathcal{T}_1 & 2 & 0 \\ \mathcal{T}_{k-1} & \mathcal{T}_{k-2} & \mathcal{T}_{k-3} & \cdots & \mathcal{T}_2 & \mathcal{T}_1 & 1 \\ \mathcal{T}_{k} & \mathcal{T}_{k-1} & \mathcal{T}_{k-2} & \cdots & \mathcal{T}_3 & \mathcal{T}_2 & \mathcal{T}_1 \end{pmatrix}\ . \tag{9}$$

For example, again with $\mathcal{E}_0 = 1$,

$$\mathcal{E}_1 = \mathcal{T}_1 \ , \tag{10}$$

$$\mathcal{E}_2 = \det \begin{pmatrix} \mathcal{T}_1 & 1 \\ \mathcal{T}_2 & \mathcal{T}_1 \end{pmatrix} = \mathcal{T}_1^2 - \mathcal{T}_2 \ , \tag{11}$$

$$\mathcal{E}_3 = \det \begin{pmatrix} \mathcal{T}_1 & 2 & 0 \\ \mathcal{T}_2 & \mathcal{T}_1 & 1 \\ \mathcal{T}_3 & \mathcal{T}_2 & \mathcal{T}_1 \end{pmatrix} = \mathcal{T}_1^3 - 3\mathcal{T}_1\mathcal{T}_2 + 2\mathcal{T}_3 \ , \tag{12}$$

$$\mathcal{E}_4 = \det \begin{pmatrix} \mathcal{T}_1 & 3 & 0 & 0 \\ \mathcal{T}_2 & \mathcal{T}_1 & 2 & 0 \\ \mathcal{T}_3 & \mathcal{T}_2 & \mathcal{T}_1 & 1 \\ \mathcal{T}_4 & \mathcal{T}_3 & \mathcal{T}_2 & \mathcal{T}_1 \end{pmatrix} = \mathcal{T}_1^4 - 6\mathcal{T}_1^2\mathcal{T}_2 + 8\mathcal{T}_1\mathcal{T}_3 + 3\mathcal{T}_2^2 - 6\mathcal{T}_4 \ , \tag{13}$$

etc.

How should one add field-dependent potentials to this hierarchy of models?

I came up with a simple scheme in collaboration with David Fairlie.

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Abstract

We show how to include field potentials in the construction of galileon hierarchies so that special kink and soliton traveling wave solutions of the underlying theory still hold true for the full hierarchy.

An interesting feature [1] of galileon models in flat spacetime [3] is that *unidirectional traveling waves* (UTWs) give solutions to the equations of motion for *all* the models in the galileon hierarchy, where UTWs are defined as fields that have the form $\phi(x) = f(u_\alpha x_\alpha)$ with constant u_α. For example, in an obvious way such traveling waves *always* satisfy the first nonlinear galileon equation of motion, $\mathcal{E}_2[\phi] = \phi_{\alpha\alpha}\phi_{\beta\beta} - \phi_{\alpha\beta}\phi_{\alpha\beta} = 0$, for any u and for any f, so long as f'' exists. Similarly, in n spacetime dimensions, all the higher galileon equations of motion ($\mathcal{E}_k[\phi] = 0$ for $3 \le k \le n$) are satisfied by UTWs given only the condition that f'' exists.

But now, in addition to the usual self-couplings that involve only derivatives of the fields, consider galileon models with self-interactions involving the fields themselves, for example in the form of polynomials of derivatives and/or fields. As is well-known — at least for non-galileon models — for special potentials elementary kink and/or soliton solutions *may* exist. What conditions are necessary or sufficient to ensure that selected elementary kink or soliton solutions of the original underlying theory are also solutions of *all* the models contained in the corresponding galileon hierarchy? Of course, to answer this question we need to know precisely what the hierarchy is when potentials are present.

Let us define the original theory by

$$A_1 = \int L_1[\phi]\ d^n x\ , \qquad L_1[\phi] = \tfrac{1}{2}\phi_\alpha\phi_\alpha - V(\phi) \tag{1}$$

$$\delta A_1 = (-)\int \mathcal{E}_1[\phi]\ \delta\phi\ d^n x\ , \qquad \mathcal{E}_1[\phi] = \phi_{\beta\beta} + V' \tag{2}$$

Also define a "flipped" Lagrangian that differs from L_1 just by flipping the sign of the potential.

$$\widetilde{L_1}[\phi] = \tfrac{1}{2}\phi_\alpha\phi_\alpha + V(\phi) \tag{3}$$

$\widetilde{L_1}$ turns out to be a useful construct. In particular, as we will discuss in detail in the following, whereas $L_1 \neq 0$ for the Goldstone kink or the elementary sine-Gordon solitons in 1+1 spacetime, it turns out that $\widetilde{L_1} = 0$ for such field configurations.

An obvious first guess for the initial step up the galileon hierarchical ladder would be by direct analogy to the situation without a potential. That is to say, let

$$A = \int L[\phi]\ d^n x \tag{4}$$

$$L[\phi] = L_1[\phi]\,\mathcal{E}_1[\phi] = \left(\tfrac{1}{2}\phi_\alpha\phi_\alpha - V(\phi)\right)\left(\phi_{\beta\beta} + V'(\phi)\right) \tag{5}$$

Equivalently,

$$L[\phi] = \tfrac{1}{2}\phi_\alpha\phi_\alpha\phi_{\beta\beta} + \tfrac{3}{2}\phi_\alpha\phi_\alpha V'(\phi) - V(\phi)\,V'(\phi) - \left(V(\phi)\,\phi_\beta\right)_\beta \tag{6}$$

where the last term is a total divergence and may be discarded when obtaining equations of motion in the bulk. It follows by direct calculation that

$$\delta A = (-)\int \mathcal{E}[\phi]\ \ \delta\phi\ d^n x \tag{7}$$

$$\mathcal{E}[\phi] = (\mathcal{E}_1[\phi] + V'(\phi))\,\mathcal{E}_1[\phi] + 3\widetilde{L_1}[\phi]\,V''(\phi) - (V'(\phi))^2 - 2V(\phi)\,V''(\phi) - \phi_{\alpha\beta}\phi_{\alpha\beta} \tag{8}$$

Or to rewrite the equations of motion in another, equivalent form,

$$\mathcal{E}[\phi] = \phi_{\alpha\alpha}\phi_{\beta\beta} - \phi_{\alpha\beta}\phi_{\alpha\beta} + 3V'(\phi)\,\mathcal{E}_1[\phi] + 3V''(\phi)\,\widetilde{L_1}[\phi] - \left(V^2(\phi)\right)'' \tag{9}$$

Unfortunately, this guess fails to preserve some interesting features of the theory based on L_1.

Specifically, for nonlinear V^2, any *elementary* UTW[1] kinks and/or solitons of the original theory are no longer solutions of the model based on L, and fail to satisfy $\mathcal{E} = 0$, because the last three terms in (8) fail to cancel, or equivalently, because the very last term in (9) does not vanish. Therefore, some further effort is required to find solutions of the modified theory. To keep UTWs in the solution set of the new theory, without any changes in the functional forms of those traveling waves, it is necessary to make additional modifications of L, hence to change $\mathcal{E}$.

One approach to these additional modifications would be to add a new term $U(\phi)$ to the purely potential part of L, i.e. $L \to L - U(\phi)$. Clearly, from (9), the required new term is just

$$U(\phi) = \frac{d}{d\phi}\ V^2(\phi) \tag{10}$$

This will increase the potential part of L by a factor of three. That is to say,

$$L[\phi] - U(\phi) = \tfrac{1}{2}\phi_\alpha\phi_\alpha\phi_{\beta\beta} + \tfrac{3}{2}\phi_\alpha\phi_\alpha V'(\phi) - 3V(\phi)\,V'(\phi) - \left(V(\phi)\,\phi_\beta\right)_\beta \tag{11}$$

The resulting equation of motion, $\mathcal{E}[\phi] + U'(\phi) = 0$, is now satisfied by any elementary UTW kink or soliton solution of the original $\mathcal{E}_1[\phi] = 0$. However, at this point it is not clear whether or not this approach can be pursued to construct and then solve additional galileon models farther up in the hierarchy.

Another approach would be to modify both the potential part and *some* of the derivative terms in L. Instead of (4) and (5), let us take

$$A_2 = \int L_2[\phi]\ d^n x \tag{12}$$

$$L_2[\phi] = \widetilde{L_1}[\phi]\,\mathcal{E}_1[\phi] = \left(\tfrac{1}{2}\phi_\alpha\phi_\alpha + V(\phi)\right)\left(\phi_{\beta\beta} + V'(\phi)\right) \tag{13}$$

The purely potential part of the Lagrangian is now

$$V_2(\phi) = -V(\phi)\,V'(\phi) = -\tfrac{1}{2}U(\phi) \tag{14}$$

Equivalently

$$L_2[\phi] = \tfrac{1}{2}\phi_\alpha\phi_\alpha\phi_{\beta\beta} - \tfrac{1}{2}\phi_\alpha\phi_\alpha V'(\phi) + V(\phi)\,V'(\phi) + \left(V(\phi)\,\phi_\beta\right)_\beta \tag{15}$$

On the one hand, this differs from the previous guess by $L_2[\phi] - L[\phi] = 2V(\phi)\left(\phi_{\beta\beta} + V'(\phi)\right)$, so terms with two derivatives as well as purely potential terms are different. The same is true when $L_2[\phi]$ is compared to the previous combination $L[\phi] - U(\phi)$, namely, $L_2[\phi] - (L[\phi] - U(\phi)) = 2V(\phi)\left(\phi_{\beta\beta} + 2V'(\phi)\right)$.

On the other hand, the action (12) is *obviously* stationary for all solutions of the original theory that satisfy $\widetilde{L_1} = 0$ as well as $\mathcal{E}_1 = 0$. This follows from the chain rule applied to first-order variations,

$$\delta A_2 = \int \left(\widetilde{L_1}[\phi]\ \ \delta\mathcal{E}_1[\phi] + \mathcal{E}_1[\phi]\ \ \delta\widetilde{L_1}[\phi]\right) d^n x \equiv (-)\int \mathcal{E}_2[\phi]\ \ \delta\phi\ d^n x \tag{16}$$

[1] *Elementary* unidirectional traveling waves of the original theory are defined here as solutions of the field equations $\mathcal{E}_1 = 0$ which *also* satisfy the condition $\widetilde{L_1} = 0$.

Since elementary UTW kink and soliton solutions of $\mathcal{E}_1 = 0$ also obey the additional condition $\widetilde{L_1} = 0$, they will remain as $\mathcal{E}_2 = 0$ solutions of the theory based on (12).

To be more explicit, direct calculation gives

$$\mathcal{E}_2\left[\phi\right] = \phi_{\alpha\alpha}\phi_{\beta\beta} - \phi_{\alpha\beta}\phi_{\alpha\beta} - V'\left(\phi\right)\mathcal{E}_1\left[\phi\right] - V''\left(\phi\right)\widetilde{L_1}\left[\phi\right] \tag{17}$$

The first two terms cancel for any traveling wave of the form $\phi\left(x\right) = f\left(u_\alpha x_\alpha\right)$ where u_α are arbitrary constants and f is any function such that f'' exists. Kinks and elementary solitons in 1+1 spacetime are precisely of this form, although colliding solitons and "breathing" configurations are not. The condition for general UTWs to solve $\mathcal{E}_2 = 0$ is then

$$V'\left(f\right)\mathcal{E}_1\left[f\right] + V''\left(\phi\right)\widetilde{L_1}\left[f\right] = 0 \tag{18}$$

In at least one other important way, this second approach is technically superior to the first.

All additional steps up the galileon hierarchical ladder may be made by the same method used in this second approach. That is to say, let

$$A_k = \int L_k\left[\phi\right]\ d^n x\ , \qquad L_k\left[\phi\right] = \widetilde{L_1}\left[\phi\right]\mathcal{E}_{k-1}\left[\phi\right] \quad \text{for } k = 2, 3, \cdots, n. \tag{19}$$

Again the chain rule shows $\mathcal{E}_k = 0$ whenever $\widetilde{L_1} = 0$ along with $\mathcal{E}_{k-1} = 0$, where

$$\delta A_k \equiv \left(-\right)\int \mathcal{E}_k\left[\phi\right]\ \delta\phi\ d^n x = \int\left(\widetilde{L_1}\ \delta\mathcal{E}_{k-1} + \mathcal{E}_{k-1}\ \delta\widetilde{L_1}\right) d^n x \tag{20}$$

Thus all original $\mathcal{E}_1 = 0$ solutions that also have $\widetilde{L_1} = 0$ are guaranteed to satisfy each and every equation of motion in the hierarchy: $\mathcal{E}_k = 0$ for $k = 1, 2, 3, \cdots, n$.

For example, consider *the Goldstone model and kink solution* in 2D spacetime, with the notation $x_\alpha = (t, x)$.

$$L_1 = \tfrac{1}{2}\left(\partial_t\phi\right)^2 - \tfrac{1}{2}\left(\partial_x\phi\right)^2 - \tfrac{1}{2}\lambda^2\left(\phi^2 - 1\right)^2 \tag{21}$$

$$\mathcal{E}_1\left[\phi\right] = \partial_t^2\phi - \partial_x^2\phi + 2\lambda^2\phi\left(\phi^2 - 1\right) \tag{22}$$

$$\widetilde{L_1} = \tfrac{1}{2}\left(\partial_t\phi\right)^2 - \tfrac{1}{2}\left(\partial_x\phi\right)^2 + \tfrac{1}{2}\lambda^2\left(\phi^2 - 1\right)^2 \tag{23}$$

$$A_1 = \int L_1\ dxdt\ , \qquad \delta A_1 = \left(-\right)\int \mathcal{E}_1\left[\phi\right]\ \delta\phi\ dxdt \tag{24}$$

The moving kink solution of $\mathcal{E}_1\left[\phi\right] = 0$ is

$$\phi_{\text{kink}}\left(x, t\right) = \tanh\theta\left(x, t\right)\ , \quad \theta\left(x, t\right) = \frac{\lambda\left(x - vt\right)}{\sqrt{1 - v^2}} \tag{25}$$

Direct calculation of the equation of motion (8) following from our initial guess in (4) then gives

$$\mathcal{E}\left[\phi_{\text{kink}}\right] = -2\lambda^4\left(7\phi_{\text{kink}}^2 - 1\right)\left(\phi_{\text{kink}}^2 - 1\right)^2 = -\frac{d}{d\phi}U\left(\phi\right)\bigg|_{\phi_{\text{kink}}} \tag{26}$$

$$U\left(\phi\right) = 2V\left(\phi\right)V'\left(\phi\right) = 2\lambda^4\phi\left(\phi^2 - 1\right)^3 \tag{27}$$

So, to repeat and emphasize our remarks surrounding (10), it is sufficient to add to L an extra potential term in order for the Goldstone kink to remain a solution. And again, to repeat earlier remarks, the addition of $U\left(\phi\right)$ merely increases the coefficient of $\phi\left(\phi^2 - 1\right)^3$ in L by a factor of three. It is quite possible that similar behavior persists for higher terms in the galileon hierarchy, given suitable additions to the various potentials in the hierarchy. However, we have not shown this.

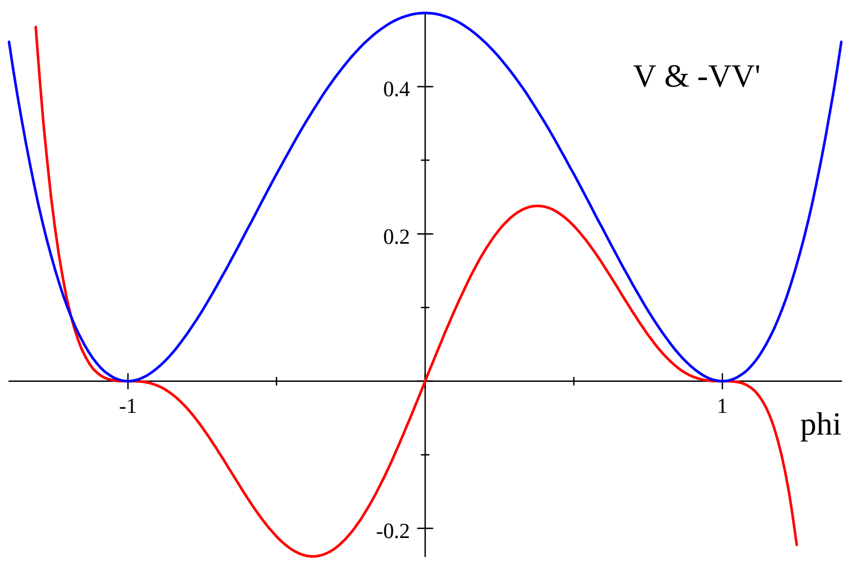
Note that U is unbounded below for this example. But then, this is true for the potential part of $L\left[\phi\right]$ before adding $U\left(\phi\right)$. Moreover, the purely derivative terms in $L\left[\phi\right]$ are trilinear in ϕ and therefore also unbounded below. So the addition of U has not changed this feature of the model.

More importantly, for the galileon potential the kink connects extrema which are inflection points, and not true minima, as opposed to the situation for the original Goldstone potential. That is to say: $\frac{d^2}{d\phi^2}\phi\left(\phi^2-1\right)^3 = 6\phi\left(7\phi^2-3\right)\left(\phi^2-1\right)\Big|_{\phi=\pm1} = 0$, as opposed to $\frac{d^2}{d\phi^2}\left(\phi^2-1\right)^2 = 12\phi^2-4\Big|_{\phi=\pm1} = 8 > 0$.

The remarks in the last two paragraphs are also appropriate for the model based on $L_2[\phi]$. In this case the potential part of the Lagrangian is

$$V_2[\phi] = -V(\phi)\,V'(\phi) = -\lambda^4\phi\left(\phi^2-1\right)^3 \tag{28}$$

Again this is unbounded below as $\phi \to +\infty$. The kink is a solution of $\mathcal{E}_2 = 0$, albeit one that connects extrema but not minima of the potential. These features are evident in the following Figure.



Comparison of $V(\phi) = \frac{1}{2}\left(\phi^2-1\right)^2$ in blue to $V_2(\phi) = -V(\phi)\,V'(\phi) = \phi\left(1-\phi^2\right)^3$ in red.

Direct calculation of $\widetilde{L_1}[\phi_{\text{kink}}]$ shows that it vanishes, since

$$\tfrac{1}{2}(\partial_t\phi)^2 - \tfrac{1}{2}(\partial_x\phi)^2 = -\tfrac{1}{2}\lambda^2\left(1-\tanh^2\theta(x,t)\right)^2 \quad \text{and} \quad \tfrac{1}{2}\lambda^2\left(\phi^2-1\right)^2 = +\tfrac{1}{2}\lambda^2\left(1-\tanh^2\theta(x,t)\right)^2 \tag{29}$$

Therefore the second approach to construct a full galileon hierarchy can be carried through, as specified in (19), with the result that ϕ_{kink} solves all the resulting equations of motion.

$$\mathcal{E}_k[\phi_{\text{kink}}] = 0 \quad \text{for} \quad 1 \le k \le n \tag{30}$$

Although $n = 2$ for the kink in 1+1 dimensional spacetime, and in that case (30) applies to only $\mathcal{E}_1$ and $\mathcal{E}_2$, nevertheless, upon replacement of $\theta(x,t) = \frac{\lambda(x-vt)}{\sqrt{1-v^2}}$ with $u_\alpha x_\alpha$, the functional form of ϕ_{kink} does provide a UTW solution to the Goldstone model equations in *any* number of dimensions. Any such UTW Goldstone solution also satisfies $\widetilde{L_1} = 0$ by essentially the same calculation as shown in (19). So the kink is a solution to (30) for any n.

The fact that $\widetilde{L_1}[\phi_{\text{kink}}] = 0$ deserves a simple explanation. It boils down to a "pseudo-particle" situation.[2] Since it is a Lorentz scalar, $\widetilde{L_1}[\phi_{\text{kink}}]$ may be evaluated by going to the rest frame of the kink, which is always possible for kinks moving *sub*luminally. In that rest frame there is no t dependence, the kink is a static configuration. But for any *static* function, $\phi(x)$, the functional $H_{\text{effective}}[\phi(x)] \equiv -\widetilde{L_1}[\phi(x)]$ is effectively a Hamiltonian for a point pseudo-particle moving in a potential $-V(\phi(x))$, with $\phi(x)$ playing the role of the particle's "coordinate" on a trajectory, and with x playing the role of the particle's "time" along that trajectory. Now, any such trajectory that solves the static equations of motion will conserve $H_{\text{effective}}$. That

[2] The reader will note that the explanation to follow is essentially the same as that used to describe instanton effects in Euclidean spacetime (http://en.wikipedia.org/wiki/Instanton). For example, see [6], [4], and references therein.

is to say, for any static solution, $-\widetilde{L_1}[\phi(x)]$ will be independent of x. The static kink $\phi_{\text{kink}}(x)$ is then just one particular trajectory as a function of the pseudo-particle time x. Evaluating $\widetilde{L_1}[\phi_{\text{kink}}(x)]$ as $x \to \pm\infty$ immediately gives $\widetilde{L_1}[\phi_{\text{kink}}(x)] = 0$. To summarize all this, the static kink represents a pseudo-particle trajectory with zero effective energy, where the pseudo-particle is moving from the left-hand peak of the inverted potential, at $\phi = -1$, to the right-hand peak, at $\phi = +1$.

Next, consider *sine-Gordon solitons*, in 2D spacetime, with the previous notation $x_\alpha = (t, x)$.

$$V(\phi) = \lambda^2 (1 - \cos\phi) \ , \qquad \mathcal{E}_1[\phi] = \partial_t^2 \phi - \partial_x^2 \phi + \lambda^2 \sin\phi \tag{31}$$

$$L_1 = \tfrac{1}{2} (\partial_t \phi)^2 - \tfrac{1}{2} (\partial_x \phi)^2 + \lambda^2 (\cos\phi - 1) \tag{32}$$

$$\widetilde{L_1} = \tfrac{1}{2} (\partial_t \phi)^2 - \tfrac{1}{2} (\partial_x \phi)^2 + \lambda^2 (1 - \cos\phi) \tag{33}$$

The elementary (winding number = 1) soliton solution to $\mathcal{E}_1[\phi] = 0$ is again a function of $\theta(x, t)$ like the Goldstone kink, but now the functional form is different.

$$\phi_{\text{soliton}}(x, t) = 4 \arctan(\exp\theta(x, t)) \ , \quad \theta(x, t) = \frac{\lambda(x - vt)}{\sqrt{1 - v^2}} \tag{34}$$

Note, for $\lambda > 0$, $\phi_{\text{soliton}}(-\infty, t) = 0$ and $\phi_{\text{soliton}}(+\infty, t) = 2\pi$ for any fixed t.

Once more we find by direct calculation that not only $\mathcal{E}_1[\phi_{\text{soliton}}] = 0$ but also

$$\widetilde{L_1}[\phi_{\text{soliton}}] = 0. \tag{35}$$

Thus ϕ_{soliton} again solves all the resulting equations of motion in the galileon hierarchy as defined by (19) and (20).

$$\mathcal{E}_k[\phi_{\text{soliton}}] = 0 \quad \text{for} \quad 1 \le k \le n \tag{36}$$

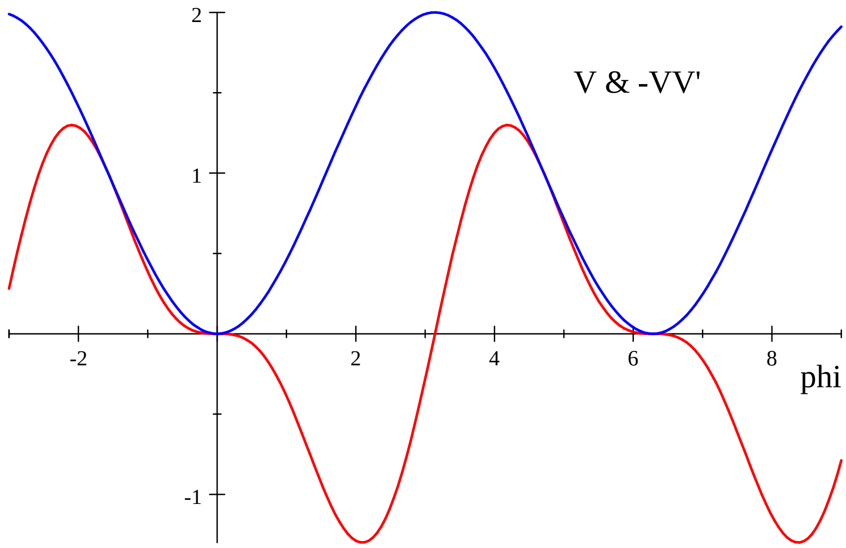
The restriction to $n = 2$ for the model in 1+1 spacetime may also be lifted, as it was for the Goldstone model, if $\theta(x, t) = \frac{\lambda(x - vt)}{\sqrt{1 - v^2}}$ is replaced with $u_\alpha x_\alpha$ to obtain a UTW with $\widetilde{L_1} = 0$ in an n dimensional spacetime.

As in the previous example, there is a pseudo-particle explanation of the null result, $\widetilde{L_1}[\phi_{\text{soliton}}] = 0$. In fact, the explanation is easily seen to apply to all elementary, non-colliding, non-breathing sine-Gordon solitons with *any* winding number N by going to the rest frame of the configuration. In that rest frame there is no time dependence, the solitons are static configurations, $\phi_N(x)$, so $H_{\text{effective}} = -\widetilde{L_1}$ is once again an effective Hamiltonian for a point pseudo-particle moving in a potential $-V$, with x the pseudo-time. For any winding number the static soliton $\phi_N(x)$ is then a pseudo-particle trajectory, as a function of the pseudo-time, such that $H_{\text{effective}}$ is conserved. Moreover, $\phi_N(x)$ represents a trajectory with zero effective energy, since the trajectory always terminates on peaks of the inverted potential $-V = \lambda^2 (\cos\phi - 1)$. Hence $\widetilde{L_1}[\phi_N(x)] = 0$.

From (15) the relevant potential terms in L_2 are

$$V_2(\phi) = -V(\phi) V'(\phi) = \lambda^4 (\cos\phi - 1) \sin\phi = \lambda^4 \left(\frac{1}{2} \sin(2\phi) - \sin\phi \right) \tag{37}$$

So, as was true for the kink, the soliton connects extrema but not absolute minima of V_2. This is evident in the following graph. As before, the extrema connected by the soliton at $\phi = 0$ and $\phi = 2\pi$ are inflection points of V_2, as opposed to being true minima for the original sine-Gordon potential $V(\phi)$ in (31).



Comparison of $V(\phi) = 1 - \cos\phi$ in blue to $V_2(\phi) = -V(\phi)\, V'(\phi) = (\cos\phi - 1)\sin\phi$ in red.

However, unlike the Goldstone case, V_2 for the sine-Gordon galileon is bounded below, and there *are* absolute minima. These are determined by

$$0 = \frac{d}{d\phi}\left((1-\cos\phi)\sin\phi\right) = -2\cos^2\phi + \cos\phi + 1 \tag{38}$$

The roots are the $\cos\phi = 1$ extrema/inflexion points connected by the original sine-Gordon soliton, the absolute *maxima* given by $\cos\phi = -\frac{1}{2}$, i.e. $\phi \in \left\{-\frac{2}{3}\pi + 2\pi k \mid k \in \mathbb{Z}\right\}$, and also the absolute *minima* given by $\cos\phi = -\frac{1}{2}$, i.e. $\phi \in \left\{\frac{2}{3}\pi + 2\pi k \mid k \in \mathbb{Z}\right\}$, at which points

$$V_2\left(\phi = \pm\frac{2\pi}{3}\right) = \left.(\cos\phi - 1)\sin\phi\right|_{\phi=\pm\frac{2}{3}\pi} = \mp\frac{3\sqrt{3}}{4} \approx \mp 1.299 \tag{39}$$

Simple physical arguments imply that there are new solitons that connect these absolute minima, and it is not difficult to compute solutions numerically. However, although there are explicit integral forms for $x(\phi)$ following from energy conservation, we have not found sufficiently simple, closed, analytic forms for $\phi(x)$ that warrant being displayed here.

A natural question to ask about the sine-Gordon galileon system is whether it is integrable, in 1+1 spacetime, like the original sine-Gordon model. One way to understand the classical integrability of the original system is through Bäcklund transformation techniques. A generating function for an "auto-canonical" transformation of the original sine-Gordon system is given by

$$F[\phi,\psi] = \int \left[\psi\partial_x\phi + 2\lambda e^{z}\cos\left(\frac{\phi-\psi}{2}\right) - 2\lambda e^{-z}\cos\left(\frac{\phi+\psi}{2}\right)\right] dx \tag{40}$$

where z is an arbitrary parameter. This generates first-order nonlinear equations in the canonical way

$$\phi_t \equiv \frac{\delta}{\delta\phi}F[\phi,\psi] = -\psi_x - \lambda e^{z}\sin\left(\frac{\phi-\psi}{2}\right) + \lambda e^{-z}\sin\left(\frac{\phi+\psi}{2}\right) \tag{41}$$

$$\psi_t \equiv -\frac{\delta}{\delta\psi}F[\phi,\psi] = -\phi_x - \lambda e^{z}\sin\left(\frac{\phi-\psi}{2}\right) - \lambda e^{-z}\sin\left(\frac{\phi+\psi}{2}\right) \tag{42}$$

Consistency of these first-order equations implies that both $\mathcal{E}_1[\phi] = 0$ and $\mathcal{E}_1[\psi] = 0$, for any z, as follows from taking second derivatives of (41) and (42).

For our purposes, it is useful to compare $\widetilde{L_1}[\phi]$ and $\widetilde{L_1}[\psi]$ for the ϕ and ψ solutions connected by the first-order Bäcklund equations. Direct calculation gives

$$\lambda^2 (1-\cos\phi) - \lambda^2 (1-\cos\psi) = 2\lambda^2 \sin\left(\frac{\phi-\psi}{2}\right)\sin\left(\frac{\phi+\psi}{2}\right) \tag{43}$$

$$\tfrac{1}{2}(\phi_t+\psi_x)^2 - \tfrac{1}{2}(\phi_x+\psi_t)^2 = -2\lambda^2 \sin\left(\frac{\phi-\psi}{2}\right)\sin\left(\frac{\phi+\psi}{2}\right) \tag{44}$$

Adding these two relations gives a useful Lemma for a pair of solutions connected by the Bäcklund transformation:

$$\begin{aligned}\widetilde{L_1}[\phi] - \widetilde{L_1}[\psi] &= -\phi_t\psi_x + \phi_x\psi_t \\ &= -\varepsilon^{\alpha\beta}\phi_\alpha\psi_\beta \quad \text{with } \varepsilon^{01} = +1 = -\varepsilon^{10}\end{aligned} \tag{45}$$

The RHS is a total divergence. Note that this is *not* true for $L_1[\phi] - L_1[\psi]$. (However, it is true that the energy-momentum tensors $\Theta_{\alpha\beta}$ for the two solutions differ by a total divergence, provided that the $\Theta_{\alpha\beta}$ are amended by conformal improvements for two spacetime dimensions, as is well-known. For example, see [2] and references therein.)

But more importantly for our purposes, the RHS of (45) will vanish if ϕ and ψ are both traveling waves, moving in the same direction at the same speed v. In particular, in a mutual rest frame for two solutions, $\phi_t(x)\psi_x(x) = 0$.

Now, by integrating the first-order Bäcklund equations for a soliton with winding number N, another soliton may be obtained with winding number $N \pm 1$. If the parameter z is chosen properly, the winding number will increase by one if the equations are integrated for a right-moving soliton, to obtain another right-moving soliton with higher winding number, *and* with the same speed. A corresponding statement is true if the initial solution is left-moving. Since $\widetilde{L_1}[\phi_{\pm1}] = 0$ (or what is even easier to see, $\widetilde{L_1}[\phi = 0] = 0$), the Lemma implies that $\widetilde{L_1}[\phi_N] = 0$ for all elementary solitons with nonzero integer winding numbers N.

In accordance with our previous discussion, it therefore follows yet again that the elementary sine-Gordon solitons ϕ_N are solutions to $\mathcal{E}_k[\phi] = 0$ for the full galileon hierarchy defined as in (19) by $L_k[\phi] = \widetilde{L_1}[\phi]\,\mathcal{E}_{k-1}[\phi]$.

On the other hand, if the RHS of (45) does not vanish, then either one or both of the solutions related by the Bäcklund transformation will fail to be a solution to $\mathcal{E}_2[\phi] = 0$. This is the situation for sine-Gordon breathing modes, and for colliding solitons with different velocities. All this considered, we are unable to say whether the L_2 model is integrable or not. But we suspect it is.

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